Statistical Properties of Wave Chaotic Scattering and Impedance Matrices

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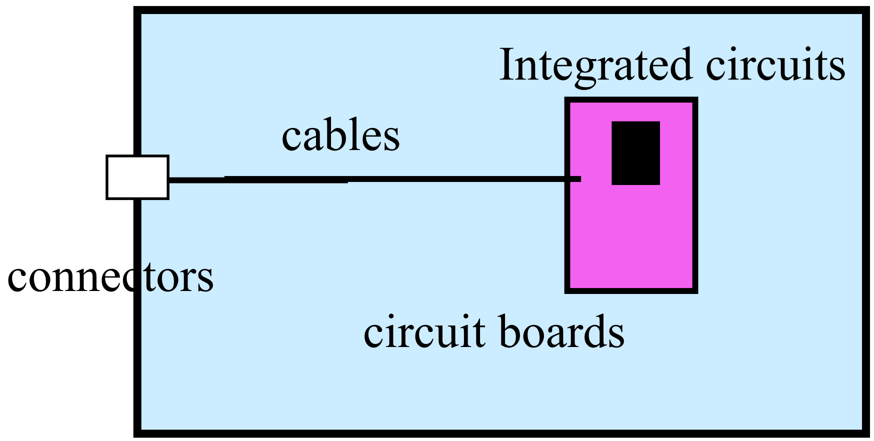
AFOSR-MURI Program Review

Electromagnetic Coupling in Computer Circuits

Schematic



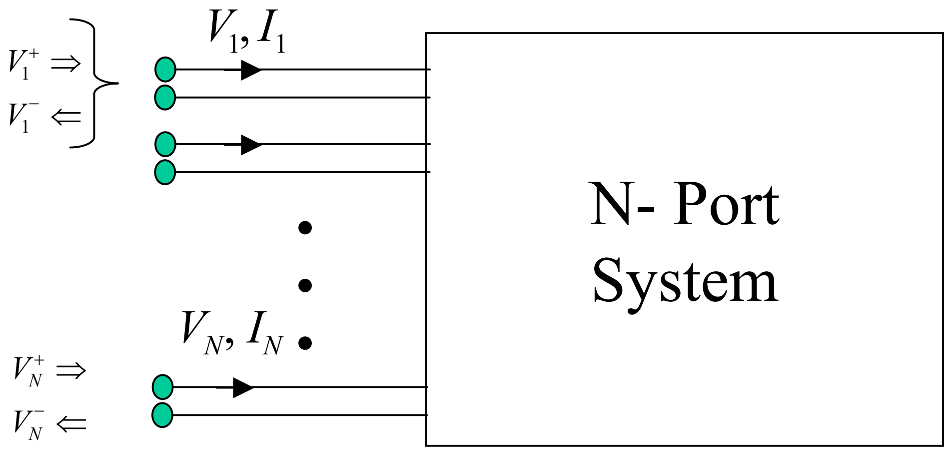
- What can be said about coupling without solving in detail the complicated EM problem ?
- **Statistical Description !**

(Statistical Electromagnetics, Holland and St. John)

- Coupling of external radiation to computer circuits is a complex processes:
  - apertures
  - resonant cavities
  - transmission lines
  - circuit elements
- Intermediate frequency range involves many interacting resonances
- System size >> Wavelength
- Chaotic Ray Trajectories

Z and S-Matrices
What is S_{ij} ?

N ports
• voltages and currents,
• incoming and outgoing waves



Z matrix

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \boldsymbol{Z} \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_2 \end{pmatrix}$$

voltage current

S matrix

$$\begin{pmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_{N1}^- \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_{N1}^+ \end{pmatrix}$$

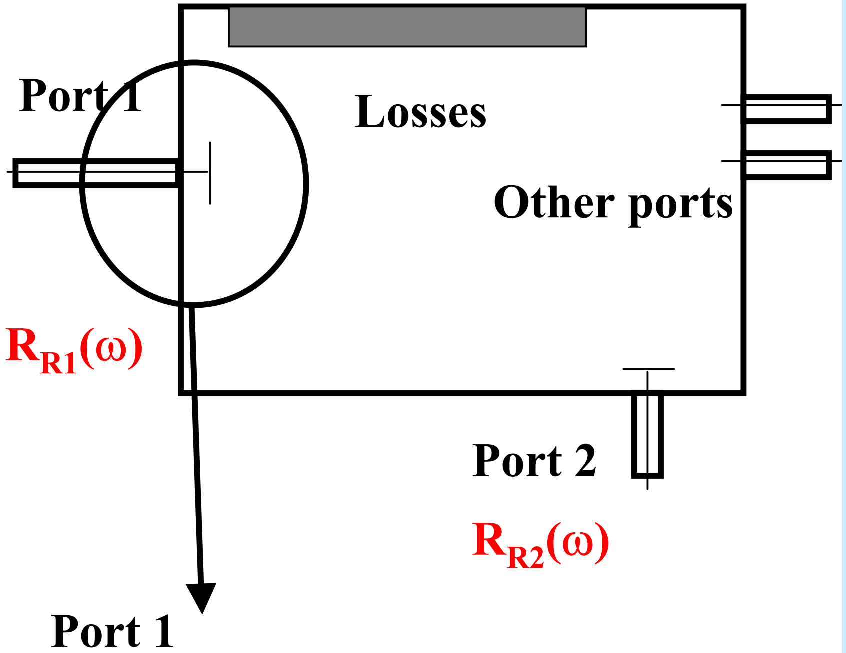
outgoing incoming

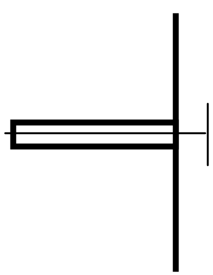
$$\boldsymbol{S} = (\boldsymbol{Z} + \boldsymbol{Z}_0)^{-1}(\boldsymbol{Z} - \boldsymbol{Z}_0)$$

$$\boldsymbol{Z}(\omega)\ ,\ \boldsymbol{S}(\omega)$$

• Complicated function of frequency
• Details depend sensitively on unknown parameters

Statistical Model of Z Matrix





Free-space radiation Resistance $R_R(\omega)$

$Z_R(\omega) = R_R(\omega) + jX_R(\omega)$

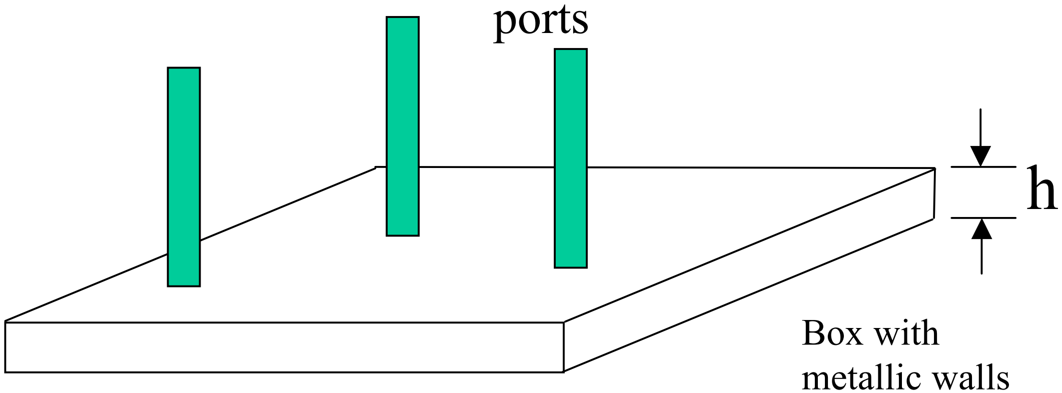
Statistical Model Impedance

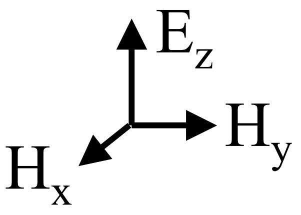
$$Z_{ij}(\omega) = -\frac{j}{\pi}\sum_n R_{Ri}^{1/2}(\omega_n) R_{Rj}^{1/2}(\omega_n) \frac{\Delta\omega_n^2 \, w_{in} w_{jn}}{\omega^2(1 + jQ^{-1}) - \omega_n^2}$$

System parameters
- **Radiation Resistance** $R_{Ri}(\omega)$
- $\Delta\omega^2_n$ **- mean spectral spacing**
- Q **-quality factor**

Statistical parameters
- ω_n **- random spectrum**
- w_{in}**- Guassian Random variables**

Two Dimensional Resonators





Only transverse magnetic (TM) propagate for $f < c/2h$

- Anlage Experiments
- Power plane of microcircuit

Voltage on top plate

$$E_z(x,y) = -V_T(x,y)/h$$

Wave Equation for 2D Cavity $E_z = -V_T/h$

- Cavity fields driven by currents at ports, (assume $e^{j\omega t}$ dependence) :

$$\nabla_\perp^2 V_T + k^2 V_T = -jkh\eta \sum_{ports} u_i I_i$$

$$\eta = \sqrt{\mu / \varepsilon}$$

$$k = \omega / c$$

Profile of excitation current

- Voltage at j[th] port:

$$V_j = \int dxdy u_j V_T$$

- Impedance matrix $Z_{ij}(k)$:

$$Z_{ij} = -jkh\eta \int u_i \left(\nabla_\perp^2 + k^2 \right)^{-1} u_j \, dxdy$$

- Scattering matrix:

$$\boldsymbol{S}(k) = \left(\boldsymbol{Z} + Z_0 \boldsymbol{I} \right)^{-1} \left(\boldsymbol{Z} - Z_0 \boldsymbol{I} \right)$$

Five Different Methods of Solution

Problem, find: $$Z_{ij} = -jkh\eta \int u_i \left(\nabla_\perp^2 + k^2\right)^{-1} u_j \, dxdy$$

1. Computational EM - HFSS

2. Experiment - Anlage, Hemmady

3. Random Matrix Theory - replace wave equation with a matrix with random elements
   No losses

4. Random Coupling Model - expand in Chaotic Eigenfunctions

5. Geometric Optics - Superposition of contributions from different ray paths
   Not done yet

Preprint available: Z_and_S_1.pdf

Expand V_T in Eigenfunctions of Closed Cavity

$$Z_{ij} = -jkh\eta \int u_i \left(\nabla_\perp^2 + k^2\right)^{-1} u_j \, dxdy = -jkh\eta \sum_{\text{modes}-n} \frac{\langle u_i \phi_n \rangle \langle \phi_n u_j \rangle}{k^2 - k_n^2}$$

Where:

1. ϕ_n are eigenfunctions of closed cavity $\quad \langle u_j \phi_n \rangle = \int dxdy \, u_j \phi_n$

2. k_n^2 are corresponding eigenvalues $\quad k_n = \omega_n / c$

***Z_{ij}* - Formally exact**

Random Coupling Model
Replace ϕ_n by Chaotic Eigenfunctions

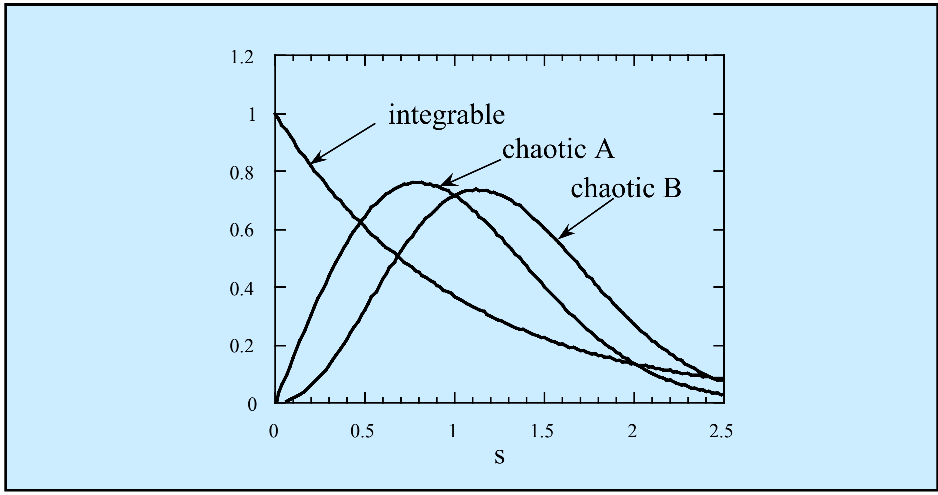
$$Z_{ij} = -jkh\eta \sum_{\text{modes}-n} \frac{\langle u_i \phi_n \rangle \langle \phi_n u_j \rangle}{k^2 - k_n^2}$$

1. Replace eigenfunction with superposition of random plane waves

$$\phi_n = \lim_{N\to\infty} \mathrm{Re}\left\{ \sqrt{\frac{2}{AN}} \sum_{k=1}^{N} a_k \exp\left[i\left(k_n \boldsymbol{e}_k \cdot \boldsymbol{x} + \theta_k \right) \right] \right\}$$

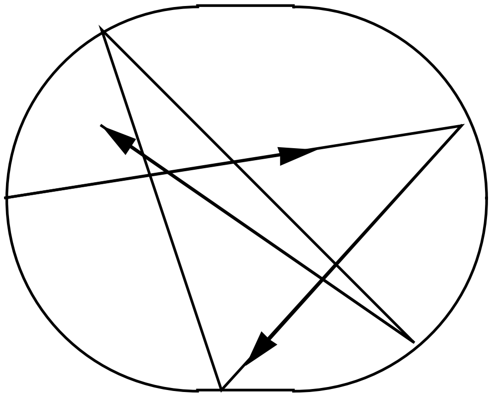
Random amplitude Random direction Random phase

2. Eigenvalues k_n^2 are distributed according to appropriate statistics



Normalized Spacing $s_n = (k_{n+1}^2 - k_n^2) / \langle \Delta k^2 \rangle$

Chaotic Eigenfunctions



Rays ergodically fill phase space.

Eigenfunctions appear to be a superposition of plane wave with random amplitudes and phases.

$$\phi_n = \lim_{N\to\infty} \mathrm{Re}\left\{\sqrt{\frac{2}{AN}}\sum_{j=1}^{N} a_j \exp\left[i\left(k_j \cdot x + \theta_j\right)\right]\right\}$$ Time reversal symmetry

$$\phi_n = \lim_{N\to\infty} \frac{1}{\sqrt{2AN}}\sum_{j=1}^{N} a_j \exp\left[i\left(k_j \cdot x + \alpha_j\right)\right]$$ Time reversal symmetry broken

k_j uniformly distributed on a circle $|k_j|=k_n$

ϕ_n is a Gaussian random variable

TRS

$$P(\phi) \approx \exp\left[-|\phi|^2 / 2\left\langle|\phi|^2\right\rangle\right]$$

Statistical Model for Impedance Matrix

$$Z_{ij}(k) = -\frac{j}{\pi}\sum_n R_{Ri}^{1/2}(k_n) R_{Rj}^{1/2}(k_n) \frac{\Delta k_n^2 \, w_{in} w_{jn}}{k^2(1 - jQ^{-1}) - k_n^2}$$

System parameters

$$R_{Ri}(k) = \frac{k\eta}{4}\oint \frac{d\theta_k}{2\pi}\left|\bar{u}_i(\boldsymbol{k})\right|^2 = \mathrm{Re}\{Z_{Ri}\}$$

-Radiation resistance for port i

$\Delta k^2_n = 1/(4A)$ **- mean spectral spacing**

Q **-quality factor**

Statistical parameters

w_{in}**- Guassian Random variables**

k_n **- random spectrum**

Predicted Properties of Z_{ij}

- Mean and fluctuating parts: $$Z_{ij} = \langle Z_{ij} \rangle + \delta Z_{ij}$$

- Mean part:
  (no losses)

$$\langle Z_{ii} \rangle = jX_{R,i}(k)$$

Radiation reactance

$$X_{R,i}(k) = \mathrm{Im}\{Z_R\}$$

$$\langle Z_{ij} \rangle = 0, \quad i \neq j$$

- Fluctuating part: Lorenzian distribution
  -width radiation resistance R_{Ri}

$$P(X_{ii}) = \frac{R_{Ri}}{\pi\left(X_{ii}^2 + R_{Ri}^2\right)}$$

Numerical Test of Z_{ii}

Numerically Generated Reactance $\xi = -\frac{1}{\pi}\sum_{n=1}^{N}\frac{w_n^2}{k^2 - k_n^2}$ $\qquad \xi = \bar{\xi} + \delta\xi$

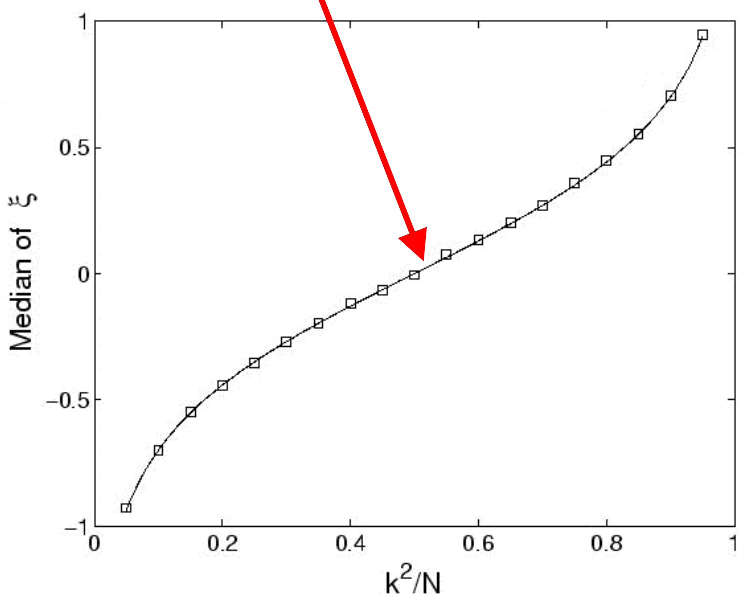
Mean $\bar{\xi}$

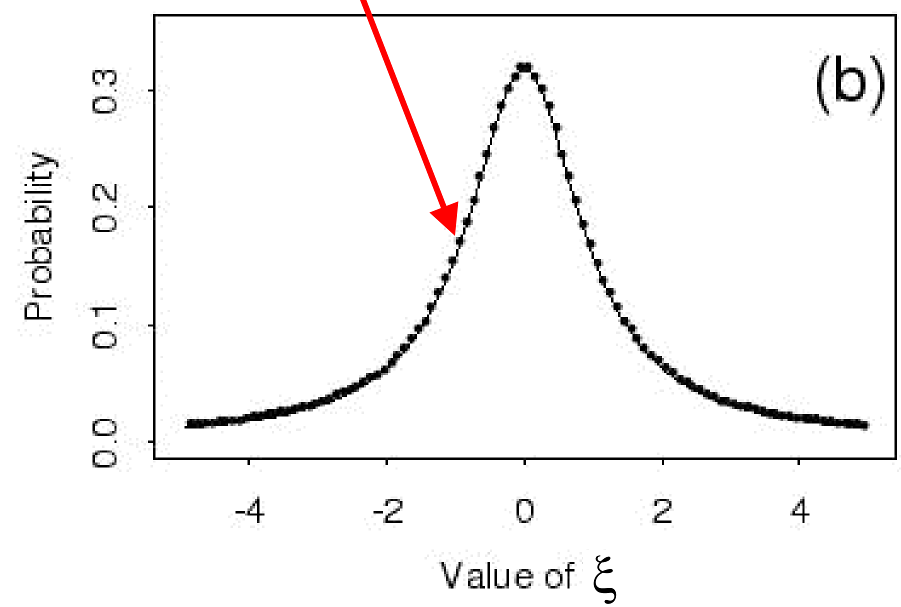
Theory: $\bar{\xi} = \frac{1}{\pi}\ln\left(\frac{N - k^2}{k^2}\right)$

Fluctuation $\delta\xi$

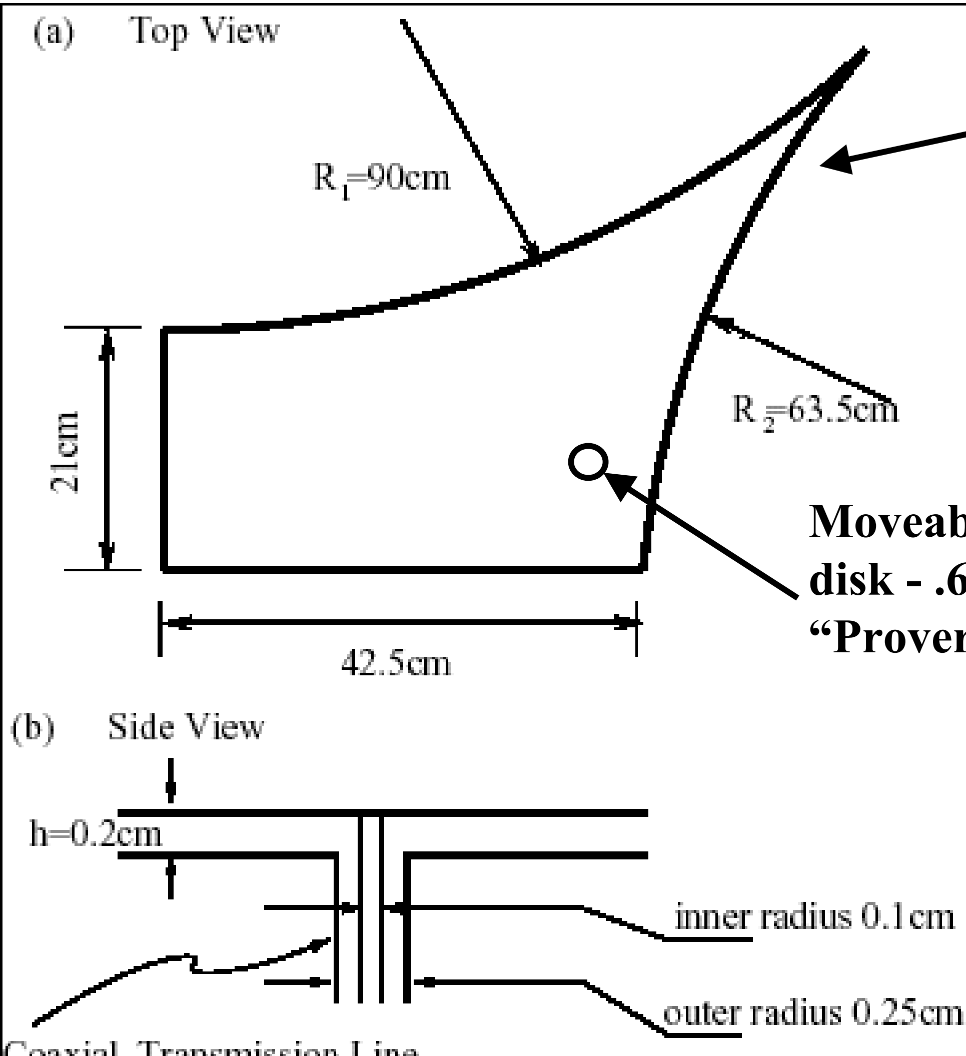
$$P(\delta\xi) = \frac{1}{\pi(1 + \delta\xi^2)}$$

10^6 realizations, N=2000,





HFSS - Solutions
Bow-Tie Cavity



Curved walls guarantee all ray trajectories are chaotic

Moveable conducting disk - .6 cm diameter
“Proverbial soda can”

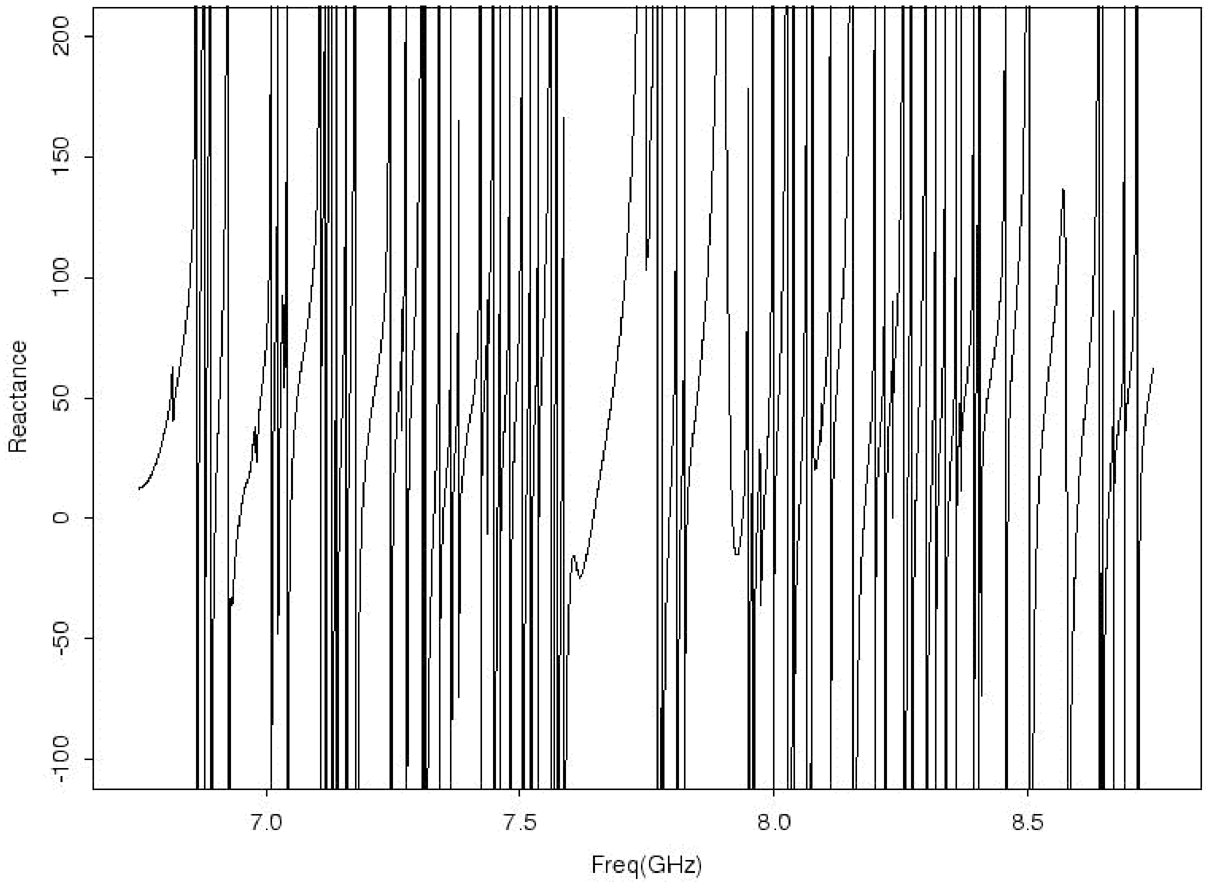
Cavity impedance calculated for
100 locations of disk
4000 frequencies
6.75 GHz to 8.75 GHz

Frequency Dependence of Reactance for a Single Realization

Mean spacing δf ≈ .016 GHz

Ω

$Z_{cav}=jX_{cav}$

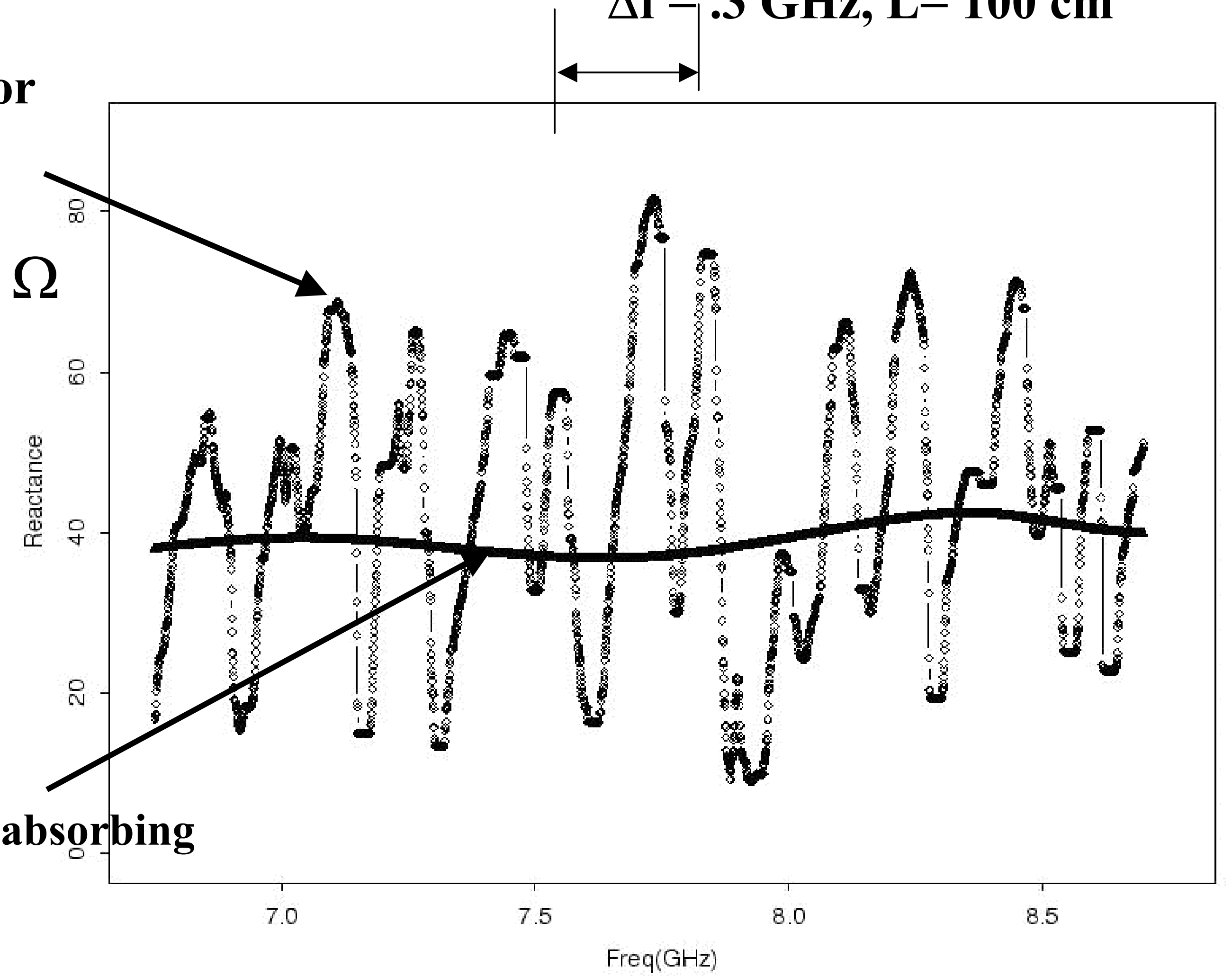


Frequency Dependence of Median Cavity Reactance

Δf = .3 GHz, L= 100 cm

Median Impedance for 100 locations of disc

Effect of strong Reflections ?

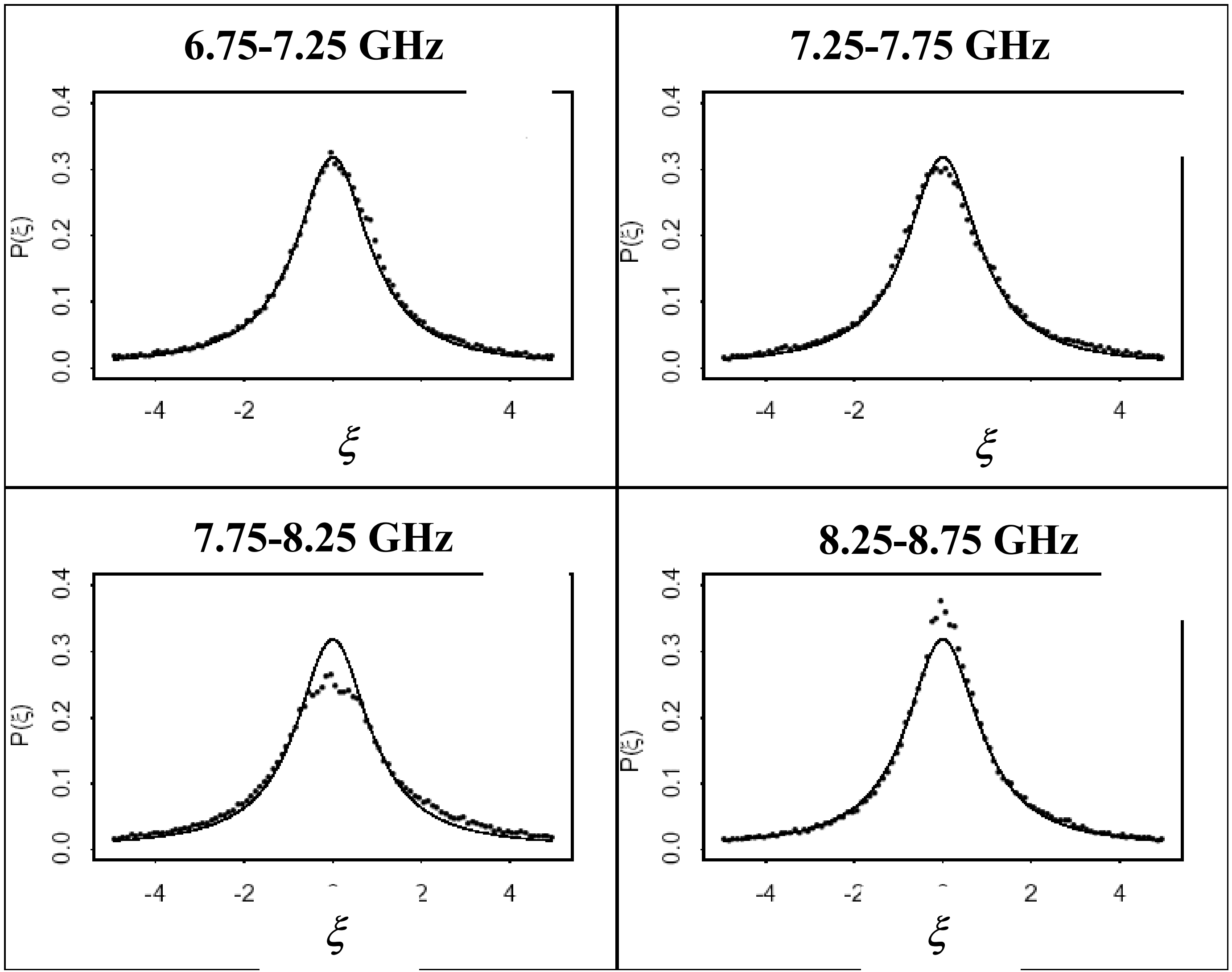


Radiation Reactance
HFSS with perfectly absorbing
Boundary conditions

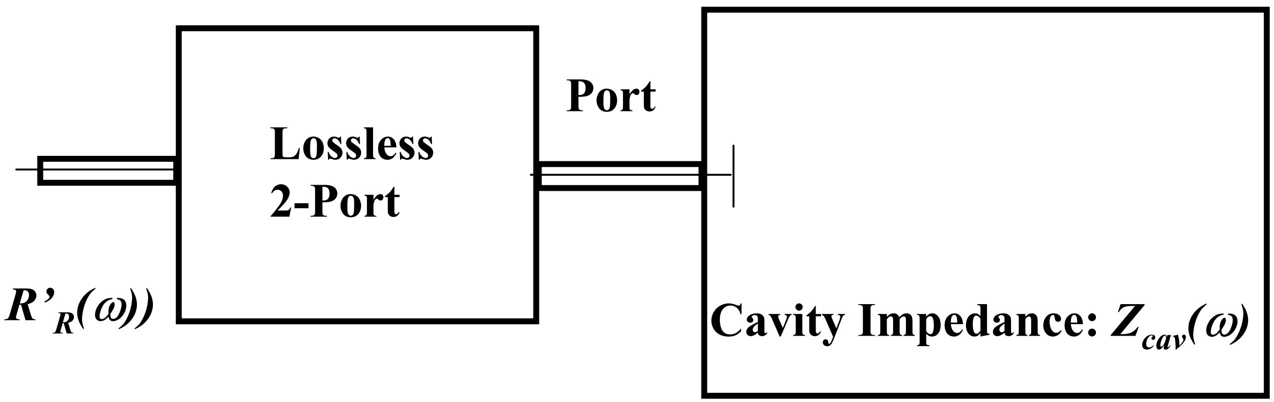
Distribution of Fluctuating Cavity Impedance

$$\xi = (X - X_R(\omega)) / R_R(\omega)$$

$R_{fs} \approx 35\ \Omega$



Impedance Transformation by Lossless Two-Port

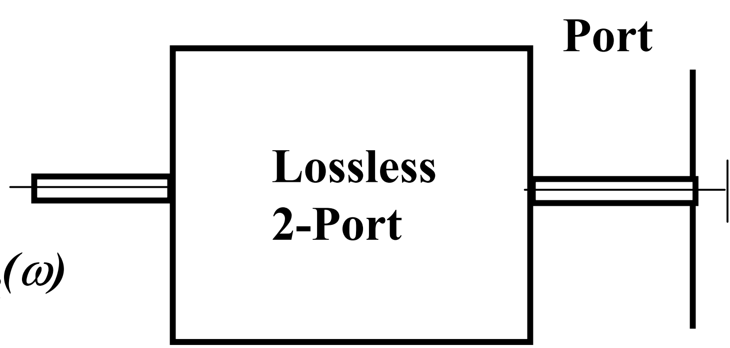


$Z'_{cav}(\omega) = j(X'_R(\omega)+\xi R'_R(\omega))$

Unit Lorenzian

$Z_{cav}(\omega) = j(X_R(\omega)+\xi R_R(\omega))$

Unit Lorenzian



$Z'_R(\omega) = R'_R(\omega)+jX'_R(\omega)$

Free-space radiation Impedance

$Z_R(\omega) = R_R(\omega)+jX_R(\omega)$

Properties of Lossless Two-Port Impedance

Eigenvalues of Z matrix

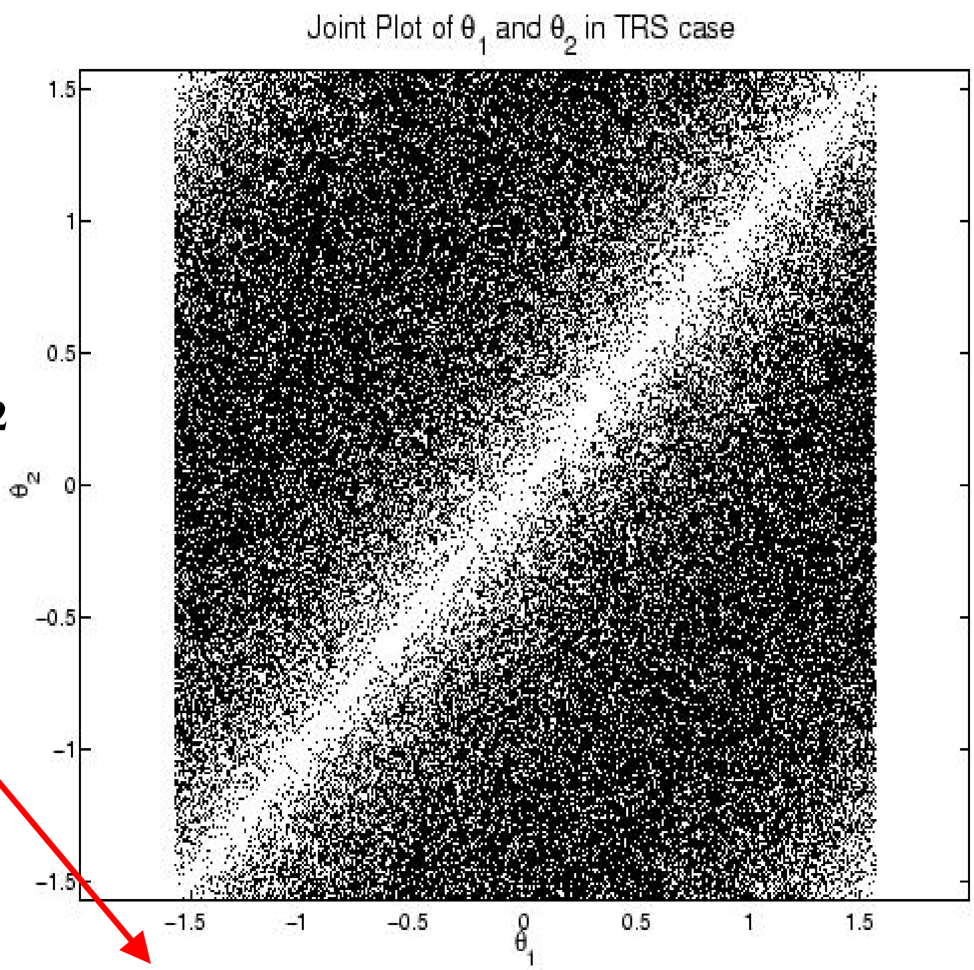
$$\det\left|\boldsymbol{Z} - jX\boldsymbol{1}\right| = 0$$

$$X_{1,2} = X_R + \xi_{1,2} R_R$$

$$\xi_{1,2} = \tan\left(\frac{\theta_{1,2}}{2}\right)$$

θ_2

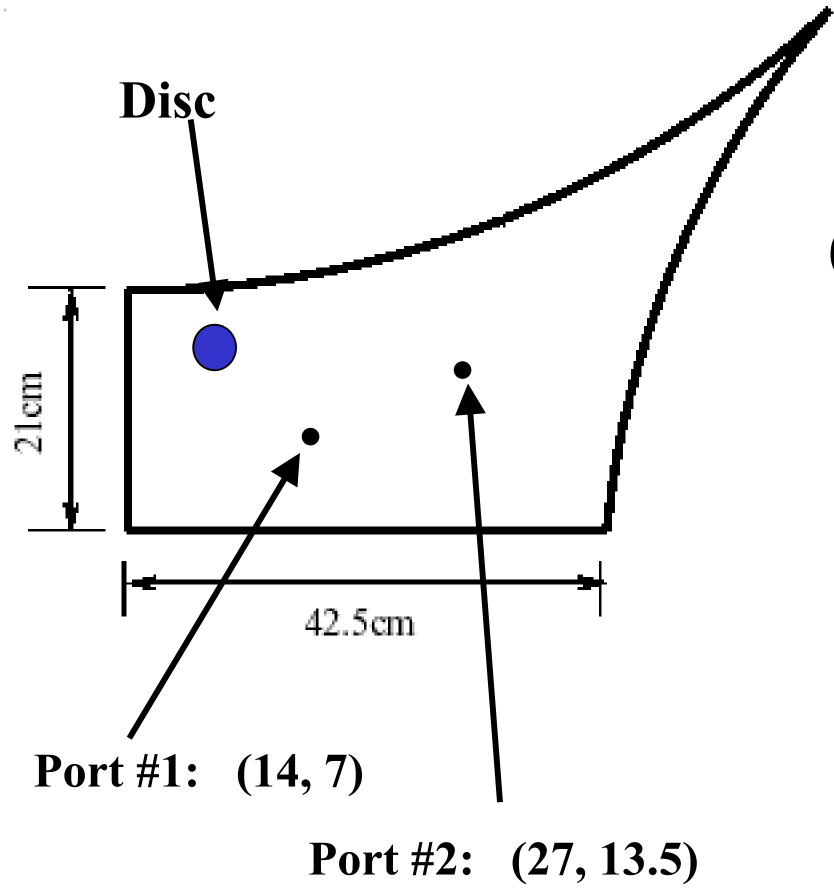
θ_1



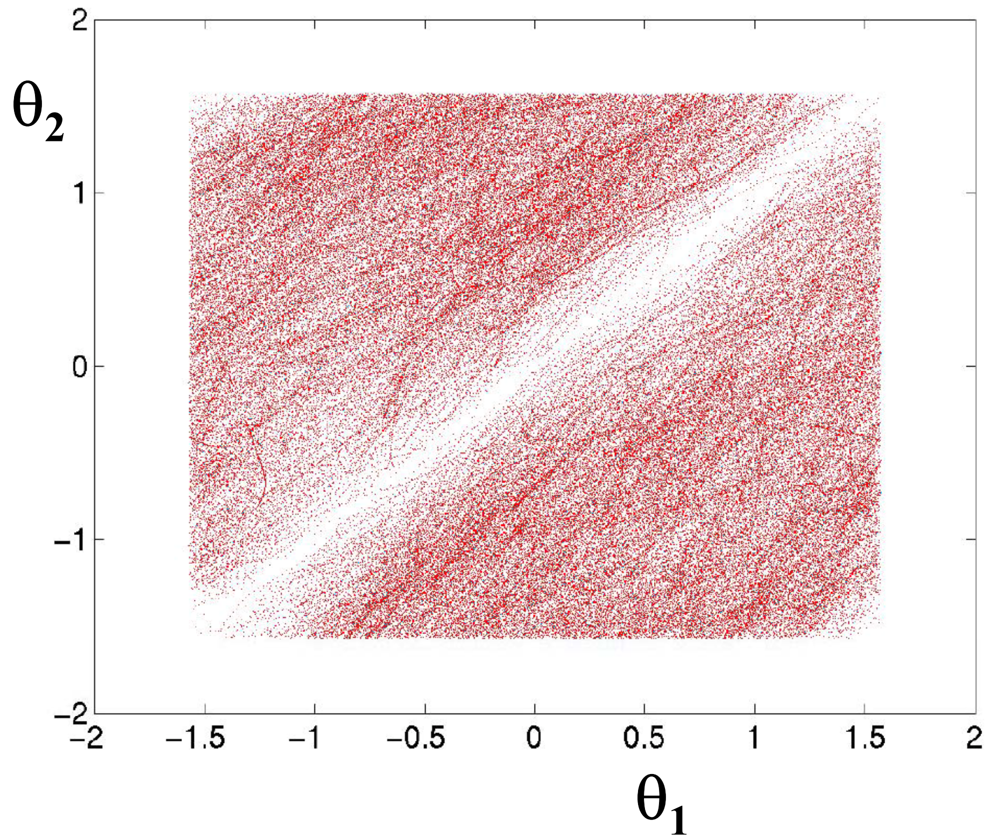
Individually $\xi_{1,2}$ are Lorenzian distributed

Distributions same as In Random Matrix theory

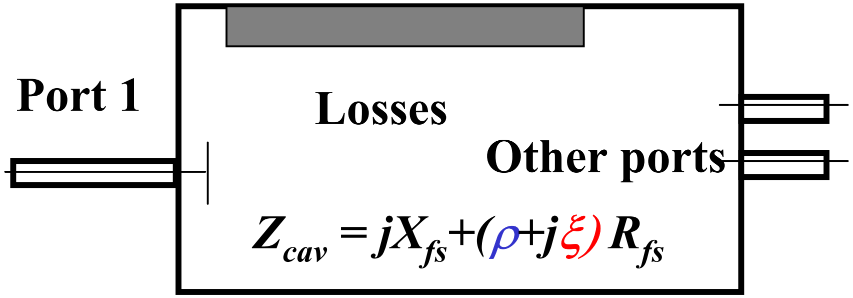
HFSS Solution for Lossless 2-Port



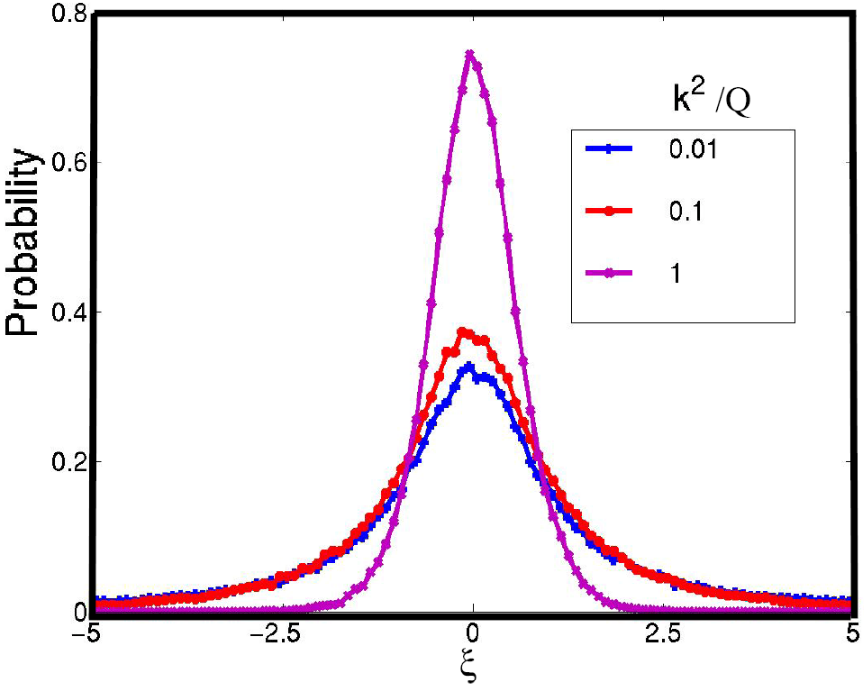
Joint Pdf for θ_1 and θ_2



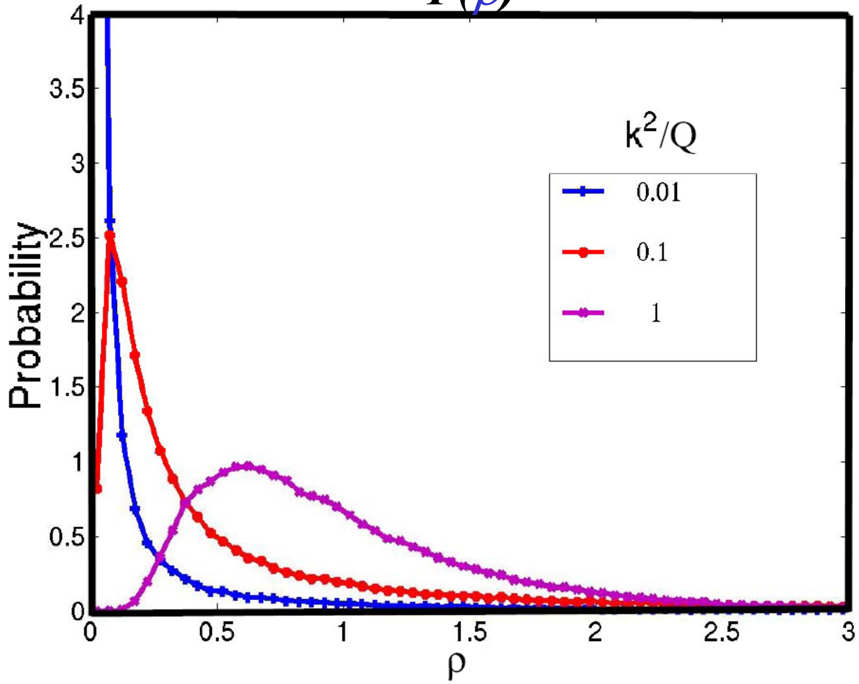
Effect of Losses



Distribution of reactance fluctuations
P(ξ)



Distribution of resistance fluctuations
P(ρ)

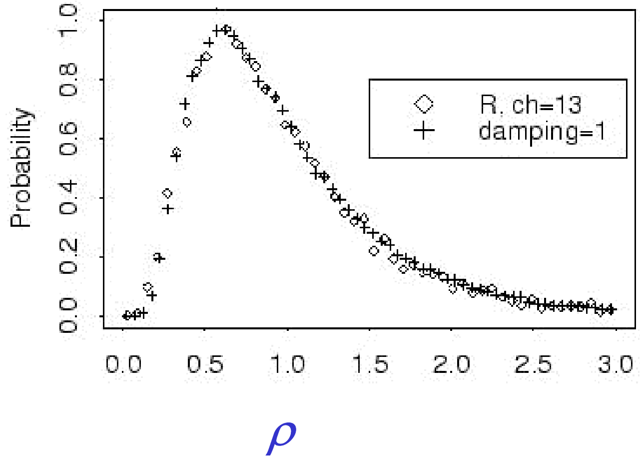


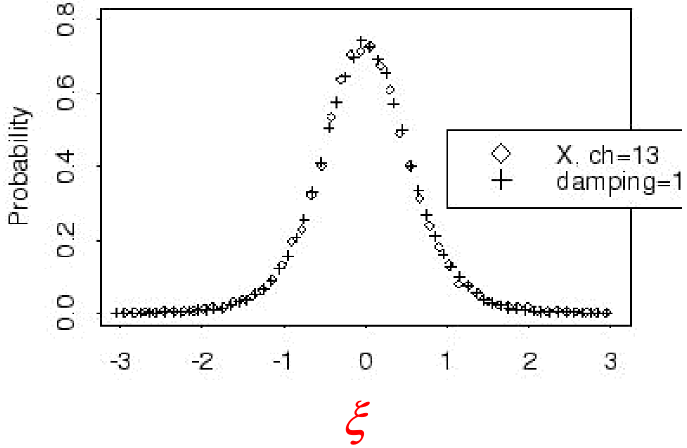
Equivalence of Losses and Channels

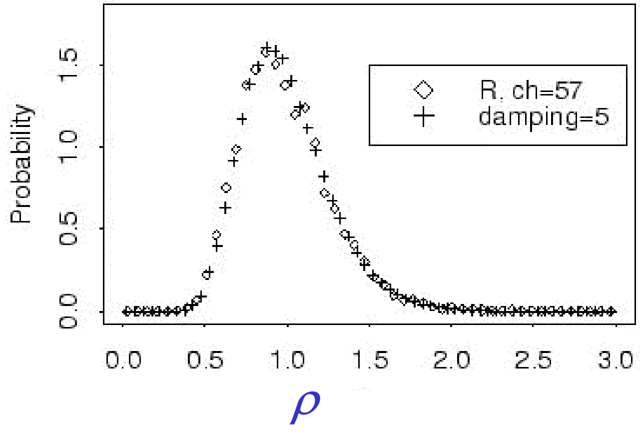
$$Z_{cav} = jX_R + (\rho + j\xi)\, R_R$$

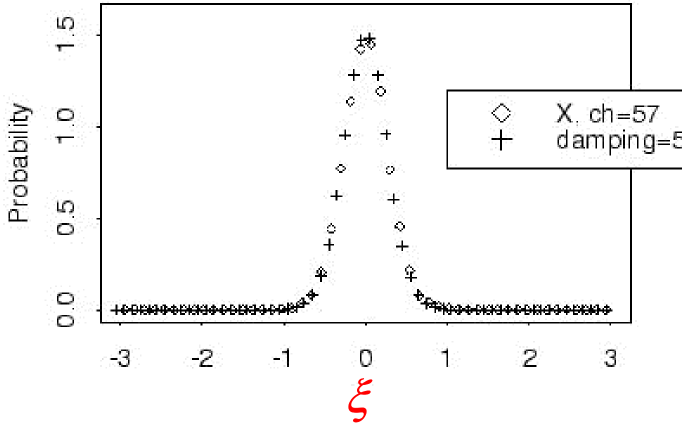
Distribution of resistance fluctuations $P(\rho)$

Distribution of reactance fluctuations $P(\xi)$









Future Directions

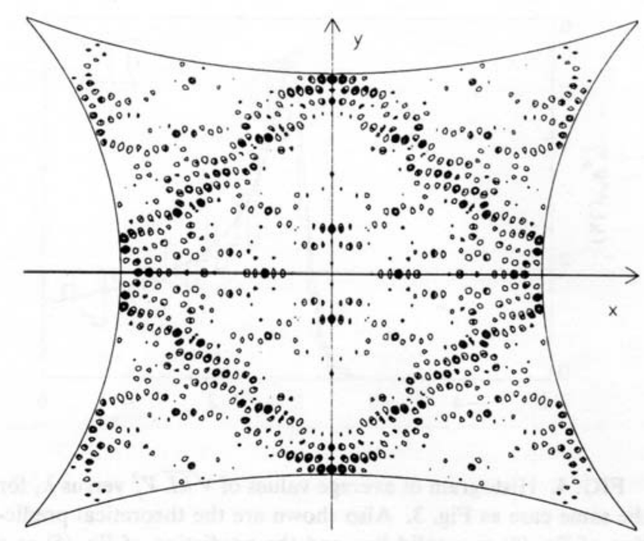
- Direct comparison of random coupling model with
  - -random matrix theory ✓
  - -HFSS solutions ✓
  - -Experiment ✓

- Exploration of increasing number of coupling channels ✓

- Study losses in HFSS
- 3D examples
- Role of Scars on low period orbits
- Generalize to systems consisting of circuits and fields

Role of Scars?

- Eigenfunctions that do not satisfy random plane wave assumption

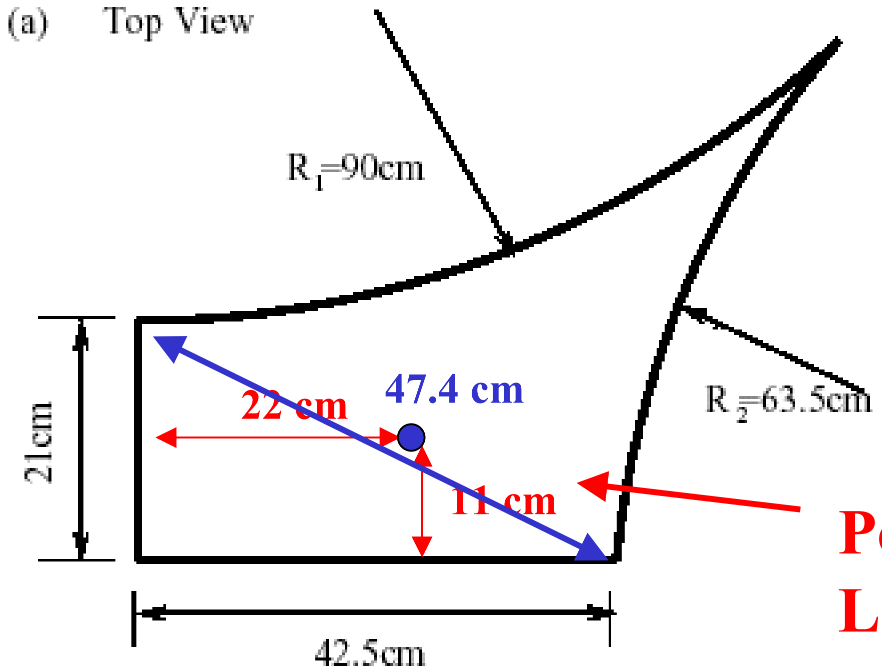


Bow-Tie with diamond scar

- Scars are not treated by either random matrix or chaotic eigenfunction theory

- Semi-classical methods

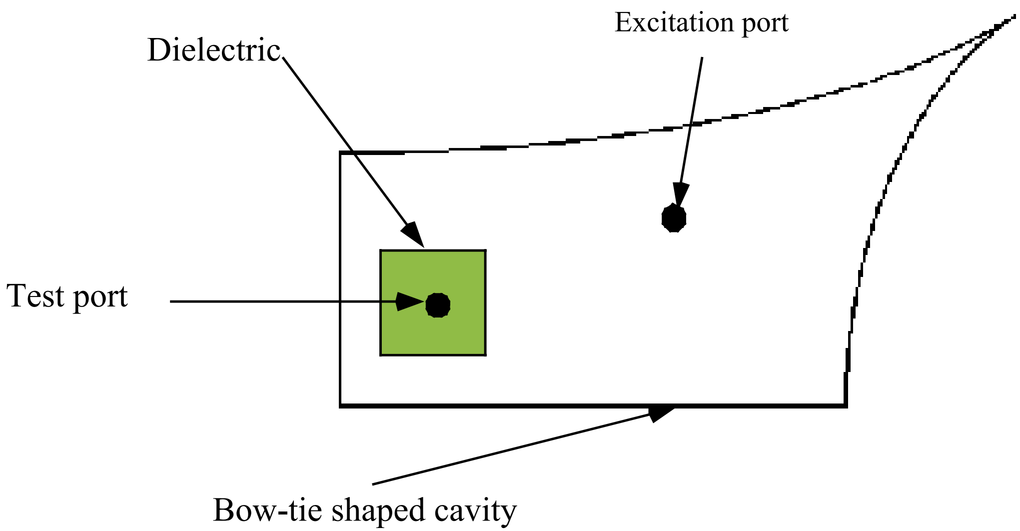
Large Contribution from Periodic Ray Paths ?



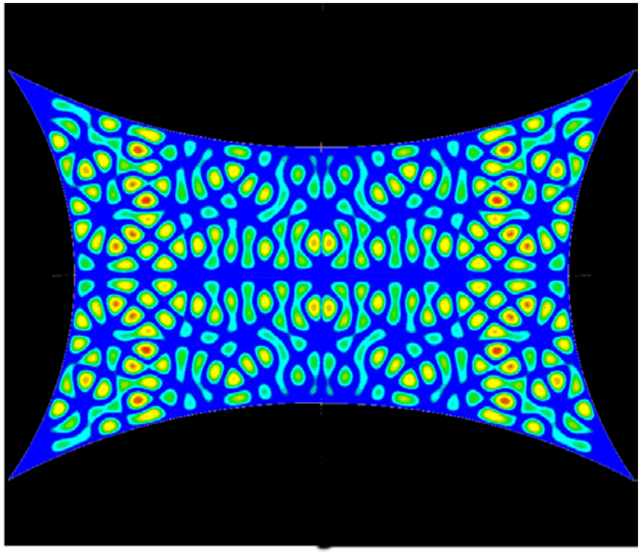
Possible strong reflections

L = 94.8 cm, Δf =.3GHz

Future Directions



- Can be addressed
  - -theoretically
  - -numerically
  - -experimentally



HFSS simulation courtesy J. Rodgers

Features:

Ray splitting

Losses

Additional complications to be added later