Universal Impedance Fluctuations in Wave Chaotic Systems

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Abstract

We experimentally investigate predictions of universal impedance fluctuations in wave chaotic systems using a microwave analog of a quantum chaotic infinite square well potential. The Random Coupling Model (RCM) treats wave and quantum chaotic systems and their associated coupling ports based on the random plane wave approximation for the eigenmodes. Specific predictions of the RCM that we test include the probability distribution functions (PDFs) of the real and imaginary parts of the normalized cavity impedance, the equality of the variances of these PDFs, the relationship between the variances and typical enclosure quality factors, the dependence of the universal PDFs on loss, and the insensitivity of the PDFs to system details. We find excellent agreement between the statistical data and the predictions of the RCM.

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There is interest in the small wavelength behavior of quantum (wave) systems whose classical (ray orbit) limit is chaotic. Despite their apparent complexity, quantum chaotic systems have remarkable universal properties. Much prior work has focused on identifying the universal statistical properties of wave chaotic systems such as quantum dots and atomic nuclei [1-3]. For example, the nearest neighbor energy level spacing statistics of these systems have universal distributions that fall into one of three classes, depending on the existence or absence of time-reversal symmetry and symplectic properties. Likewise the eigenfunctions of wave chaotic systems have universal statistical properties, such as one-point and two-point statistical distribution functions [4-6]. Here, we extend these universal statistical properties in a new direction and experimentally investigate the universality of the complex impedance matrix (and scattering matrix) fluctuations of wave chaotic systems.

We consider wave systems in the semiclassical limit consisting of enclosures that show chaos in the ray limit, but which are also coupled to their surroundings through a finite number of leads or ports. Examples include quantum dots together with their leads, wave chaotic microwave or acoustical cavities together with their coupling ports, or scattering experiments on nuclei. The statistical properties of two-port conductance fluctuations in quantum dot transport have been studied in the past. However, there have been only a few systematic investigations of the statistical properties of scattering (or impedance) matrices in quantum chaotic systems [7], concentrating mainly on the scalar reflection and transmission coefficients of open systems [8-12], and including loss [9-12]. In the related field of statistical electromagnetism [13], the statistical distribution of electromagnetic fields within complicated enclosed systems has been studied [14], but not the general properties of the scattering or impedance matrices.

In Ref. [12] (and references therein) the importance of both coupling and absorption was recognized in modifying the predictions of Random Matrix Theory (RMT) for the impedance and scattering matrices of real systems. There, the average of the reflection coefficient was measured in chaotic microwave cavities, and excellent agreement was found with the predictions of RMT, after the data was used to independently determine the coupling strength and absorption. However, the treatment of coupling and losses was based on the wave chaotic cavity data itself, which is somewhat *ad hoc* and not based on a fundamental understanding of how coupling affects the results. It would be more desirable to separate the determination of the coupling and loss information from the determination of the universal fluctuating quantities, and to do so with a minimum of theoretical complication. Also, there is only a patchwork of theoretical predictions for the statistical properties of the reflection coefficients in certain limiting cases of coupling and loss. Again it would be desirable to have global predictions for the statistical properties of wave chaotic systems that cover all possible values of both coupling and loss. Hence there is a need for increased understanding of the properties of real quantum and wave chaotic systems that are coupled to the environment.

A prediction of universal fluctuations of the complex impedance and scattering matrices of wave chaotic systems has emerged from the Random Coupling Model (RCM) [15]. Our purpose is to test specific predictions of the RCM using an experiment that allows contact to both the quantum chaotic and wave chaotic aspects of the problem. We use a quasi-two-dimensional chaotic microwave resonator [16] (see inset in Fig. 1(a)) to experimentally study the impedance and scattering matrices of wave chaotic cavities, including the coupling ports. Through the Helmholtz-Schrödinger analogy that holds true for two-dimensional electromagnetic cavities, our results can be extended to quantum chaotic systems, such as quantum dots. We experimentally test the key predictions of the RCM for one-port systems and find very good agreement.

The Scattering Matrix $\vec{\vec{S}}$ relates the incoming and outgoing state of a scattering process. For a N-port quantum system or microwave circuit, the N by N scattering matrix can be related to the amplitudes and phases of the incoming and outgoing waves as $\hat{b} = \vec{\vec{S}}\,\hat{a}$, where the column vector $\hat{b}$ ($\hat{a}$) represents the N complex waves leaving (entering) the system. In the experiment we concentrate on a one-port network and the S-matrix reduces to a complex reflection coefficient, S_{11}. For a microwave cavity, the $\hat{b}$ and $\hat{a}$ are the outgoing and incoming voltage waves at the plane of calibration. Wigner introduced a related quantity, the R-matrix, as an alternative method to describe scattering problems [17] in quantum mechanics. In this case, space is arbitrarily divided into two parts, a finite interior domain containing the scattering potential of interest, and the remaining external asymptotic region. The wavefunction for the particle in the interior domain is expressed as a linear combination of discrete bound states, while the particle is described by scattering states in the external asymptotic region. The R-matrix describes the boundary condition linearly relating the normal (n) derivative of the wavefunction ($\hat{\psi}$) to the wavefunction itself, at the boundary of the interior domain as $\hat{\psi} = \vec{\vec{R}}\ \partial\hat{\psi}/\partial n$. The analogous quantity in an electromagnetic system is the impedance matrix $\vec{\vec{Z}}$, relating voltages ($\hat{V}$) and currents ($\hat{I} \sim \partial\hat{V}/\partial$n) at the ports as $\hat{V} = \vec{\vec{Z}}\hat{I}$. The impedance matrix is related to the scattering matrix through $\vec{\vec{S}} = \vec{\vec{Z}}_0^{1/2}\left(\vec{\vec{Z}} + \vec{\vec{Z}}_0\right)^{-1}\left(\vec{\vec{Z}} - \vec{\vec{Z}}_0\right)\vec{\vec{Z}}_0^{-1/2}$, where $\vec{\vec{Z}}_0$ is a diagonal real matrix whose elements are the characteristic impedances of the transmission line modes connected to each port, and reduces to $Z = Z_0(1+S_{11})/(1-S_{11})$ for a one-port network.

The Random Coupling Model (RCM) is a stochastic model based upon the *a-priori* knowledge of the statistical properties of the chaotic eigenvalues and eigenfunctions of the system, and is capable of incorporating both the universal fluctuations in the impedance of such enclosures, as well as non-universal quantities that are specific to a given geometry and coupling mechanisms [15]. The primary information required is knowledge of the radiation impedance for each port (or lead) of the system as a function of energy or frequency. The radiation impedance depends only on the configuration of the coupling geometry and not on the cavity boundaries. It is experimentally accessible, system specific (hence nonuniversal), and independent of global properties of the system (e.g., chaos). For a lossless system, the RCM predicts that the impedance $Z = iX$ consists of a mean component that is equal to the imaginary part of the radiation impedance, and a universal fluctuating component whose scale is set by the real part of the radiation impedance [15]. Losses cause Z to have a real part, and fluctuations of the complex cavity impedance are described by universal probability distribution functions (PDFs) for both the real and imaginary parts that depend smoothly on loss.

The RCM suggests a normalization procedure to remove the dependence on details and non-universal properties, and to reveal the underlying intrinsic fluctuations of the wave chaotic impedance [15]. This involves the radiation impedance Z_{Rad} that is determined for the same coupling geometry by removing the sidewalls of the cavity to infinity so that no reflections come back to the port. Energy stored in the near field determines the imaginary part of Z_{Rad}, while the real part reflects the far field radiation, which, in the closed cavity case, gives rise to the fluctuations of Z_{Cav}. The normalized cavity impedance is,

$$Z_{Norm} = \frac{\mathrm{Re}[Z_{Cav}] + i(\mathrm{Im}[Z_{Cav}] - \mathrm{Im}[Z_{Rad}])}{\mathrm{Re}[Z_{Rad}]}. \qquad (1)$$

In the case of a lossless cavity, the imaginary part is expected to exhibit a Lorentzian distribution with unit width [15]. Losses manifest themselves through a finite real part of Z_{Norm} and also lead to the truncation of the tails of the Lorentzian distribution of *Im*(Z_{Norm}). The RCM also predicts that the variances of the *Re*(Z_{Norm}) and *Im*(Z_{Norm}) PDFs are equal in the limit $Q >> 1$ [15],

$$\sigma^2_{\mathrm{Re}\,Z_{Norm}} = \sigma^2_{\mathrm{Im}\,Z_{Norm}} \cong 0.36\,\frac{Q}{\tilde{k}^2}, \qquad (2)$$

in the case of time-reversal symmetric wave chaotic systems (Gaussian Orthogonal Ensemble). Here $\tilde{k}^2 = k^2/\Delta k^2$ where k is the free space wavenumber, Δk^2 is the mean spacing in k^2 eigenvalues for the closed cavity ($\Delta k^2 = 4\pi/A$ for a two-dimensional cavity of area A), and Q is the quality factor of the enclosure. The RCM makes quantitative predictions for the PDFs of Re[Z_{Norm}] and Im[Z_{Norm}] with loss.

For our experimental tests of the RCM, we use the quasi-two-dimensional chaotic microwave resonator shown in Fig 1. The height of the cavity (h = 7.87 mm) is small enough so that only TM modes exist in the cavity volume for frequencies below about 18.9 GHz [16]. The port consists of the center conductor of a coaxial cable that extends from the top lid of the cavity and makes contact with the bottom plate (Fig. 1(b) inset), injecting current into the bottom plate of the cavity. This cavity has previously demonstrated a crossover from GOE to GUE statistics in the eigenvalue spacing statistics [18] and eigenfunction statistics [19,20]. To perform ensemble averaging, two perturbations, made up of rectangular ferromagnetic solids wrapped in Al foil (dimensions 26.7 x 40.6 x 7.87 mm^3), are systematically scanned and rotated through out the volume of the cavity by means of a strong magnet that is placed outside the cavity.

As predicted by the RCM, the universal cavity impedance statistics can be drawn from S_{11} measurements of the perturbed cavity (referred to as the Cavity Case) and a measurement with identical coupling, but with the walls of the cavity removed to infinity (the Radiation Case). The latter condition is realized experimentally by placing microwave absorber (ARC tech- DD10017D, <-25 dB return loss between 6 and 12 GHz) along all the sidewalls of the cavity. An ensemble of wave chaotic cavities is obtained by recording S_{11} as a function of frequency (8001 points between 6 and 12 GHz) for 100 different positions and orientations of the perturbations within the cavity.

Fig.1 shows representative measured PDFs of the real and imaginary parts of the normalized cavity impedance defined by Eq. (1). The data is taken between 6.0 and 6.6 GHz from a cavity of height 7.87 mm, with the coupling antenna diameter of $2a$ = 0.635 mm (see Fig. 1(b)). The PDF data is overlaid with a single-parameter fit, based on the RCM, which simultaneously fits both histograms. The parameter is a measure of losses in the cavity, $\tilde{k}^2/Q$, and there is a close overlap between the RCM prediction and the experimental results with a choice of $\tilde{k}^2/Q$= 1.05. This translates to a typical Q of our cavity (A = 0.115 m^2) of about 152, which is in good agreement with Q values extracted from $S_{11}(\omega)$ measurements.

In Fig. 2 we examine the effects on the PDFs of variable loss in the cavity. The losses are incrementally increased within the cavity by placing 15.2 cm-long strips of microwave absorber along the inner walls of the cavity. The data shows that, as the losses within the cavity increase, the PDF of the normalized imaginary part of the impedance loses its long tails and begins to sharpen up, developing a Gaussian-like appearance. The normalized PDF of the real part smoothly evolves from being peaked below 1, into a Gaussian-like distribution that peaks at 1 and sharpens with increasing loss. The normalized PDFs are shown with their corresponding one-parameter fits (solid lines in Fig. 2) for $\tilde{k}^2/Q$ from the RCM. The inset in Fig. 4 shows the linear relationship of the fitting parameter $\tilde{k}^2/Q$ to the absorber perimeter ratio (α), where α is the ratio of the total length of the microwave absorbing strips within the cavity to the inner perimeter of the cavity. This demonstrates that the PDFs and variances behave as predicted by the RCM as loss in the cavity is varied.

We also test the degree of insensitivity of the universal properties of the normalized impedance PDFs to details and non-universal quantities. We take two identical cavities and change only the diameter of the coupling wire in the antenna from $2a$ = 1.27 mm to 0.635 mm. As seen in Fig. 3(a), this difference causes a noticeable change in the raw Im[Z_{Cav}] PDF. However, in agreement with the theoretical prediction, this difference essentially disappears in the PDFs of the scaled impedance Z_{Norm} as shown in Fig. 3(b).

Another prediction of the RCM concerns the variance of the PDFs and its relation to the quality factor Q. We can systematically change the Q by changing the cavity height and the amount of microwave absorber along the interior walls. Fig. 4 shows the variance of the PDFs of the real and imaginary parts of Z_{Norm} compared to the $\tilde{k}^2/Q$ fit parameter for the same PDFs, for a number of cases. The open symbols (+ symbols) are the variance of the real part of Z_{Norm} for a cavity height of h = 7.87 mm (h = 1.78 mm); the closed symbols (× symbols) are the variance of the imaginary part of Z_{Norm} for h = 7.87 mm (h = 1.78 mm). The agreement between the data and the Q >> 1 theory (Eq. (2), solid line) is very good. We note a close overlap between the variances of Re[Z_{Norm}] and Im[Z_{Norm}] consistent with the RCM prediction that they are equal. This agreement is very robust experimentally and is seen independent of frequency, type and amount of loss, cavity dimensions, antenna properties, etc.

We have tested some of the key predictions of the RCM for wave and quantum chaotic systems. These results are based on very general considerations and should apply equally well to conductance measurements through quantum dots, impedance or scattering matrix measurements on electromagnetic or acoustic enclosures, and scattering experiments from nuclei and Rydberg atoms. The RCM can be used to predict the bounds on the mean and fluctuating parts of the cavity impedance (or scattering matrix) for any particular type of wave chaotic setup. The statistical mean and variance of PDFs can be determined *a priori* by determining the radiation impedance of the coupling geometry. The mean value of the Im[Z_{Cav}] distribution is equal to the imaginary part of the radiation impedance Im[Z_{Rad}] of the same coupling geometry. The magnitude of the reactance fluctuations corresponds to the real part of the radiation impedance (Re [Z_{Rad}]) in the low loss case, and then diminishes as $\sqrt{Q}$ as losses increase. One also expects that the variances of the PDFs are a monotonically decreasing function of frequency and an increasing function of enclosure volume.

In conclusion we have examined key testable predictions of the RCM and found satisfactory agreement on all experimental issues directly related to the theory. We find that a single parameter simultaneous fit to two independent PDFs is remarkably robust and successful, and the fit parameter is physically reasonable. The normalized cavity impedance describes universal properties of the impedance matrix fluctuations that depend only on loss in the system. The RCM predicts new universal phenomena and accommodates non-universal system-specific features in its predictions.

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References:

[1] R. U. Haq, A. Pandey, and O. Bohigas, Phys. Rev. Lett. 48, 1086 (1982).

[2] see H.-J. Stöckmann, *Quantum Chaos* (Cambridge University Press, 1999), and references therein.

[3] Y. Alhassid, Rev. Mod. Phys. 72, 895 (2000).

[4] C. E. Porter and R. G. Thomas, Phys. Rev. 104, 483 (1956).

[5] H. Alt, H. –D. Gräf, H. L. Harney, R. Hofferbert, H. Lengeler, A. Richter, P. Schardt, and H. A. Weidenmüller, Phys. Rev. Lett. 74, 62 (1995).

[6] V. N. Prigodin, N. Taniguchi, A. Kudrolli, V. Kidambi, and S. Sridhar, Phys. Rev. Lett. 75, 2392 (1995).

[7] R. Blümel and U. Smilansky, Phys. Rev. Lett. 64, 241 (1990).

[8] Y. V. Fyodorov and H.-J. Sommers, J. Math. Phys. 38, 1918 (1997).

[9] E. Kogan, P. A. Mello, and H. Liqun, Phys. Rev. E 61, R17 (2000).

[10] C. W. J. Beenakker and P. W. Brouwer, Physica E 9, 463 (2001).

[11] D. V. Savin and H.-J. Sommers, Phys. Rev. E 68, 036211 (2003).

[12] R. A. Méndez-Sánchez, U. Kuhl, M. Barth, C. H. Lewenkopf, and H. -J. Stöckmann, Phys. Rev. Lett. 91, 174102 (2003).

[13] R. Holland and R. St. John, *Statistical Electromagnetics*, (Taylor and Francis, Philadelphia, 1999).

[14] D. A. Hill, M. T. Ma, A. R. Ondrejka, B. F. Riddle, M. L. Crawford, R. T. Johnk, IEEE Trans. Electromag. Compat. 36, 169 (1994).

[15] X. Zheng, T. M. Antonsen, and E. Ott, "Statistics of Chaotic Impedance and Scattering Matrices: The Random Coupling Model," preprint, 2003. Available at: http://www.ireap.umd.edu/MURI-2001/Review_14Nov03/Z_and_S_1.pdf

[16] A. Gokirmak, D.-H. Wu, J. S. A. Bridgewater and S. M. Anlage, Rev. Sci. Instrum. 69, 3410 (1998).

[17] E. P. Wigner and L. Eisenburd, Phys. Rev. 72, 29 (1947); A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30, 257 (1958).

[18] P. So, S. M. Anlage, E. Ott, and R. N. Oerter, Phys. Rev. Lett. 74, 2662 (1995).

[19] D.-H. Wu, J. S. A. Bridgewater, A. Gokirmak, and S. M. Anlage, Phys. Rev. Lett., 81, 2890 (1998).

[20] S.-H. Chung, A. Gokirmak, D.-H. Wu, J. S. A. Bridgewater, E. Ott, T. M. Antonsen, and S. M. Anlage, Phys. Rev. Lett. 85, 2482 (2000).

Figure Captions

Fig. 1. PDFs for the (a) real and (b) imaginary parts of the normalized cavity impedance Z_{Norm} for a wave chaotic microwave cavity between 6.0 and 6.6 GHz with h = 7.87 mm and $2a$ = 0.635 mm. Also shown is a single parameter simultaneous fit for both PDFs with $\tilde{k}^2/Q$ = 1.05. The inset in (a) shows the shape and size of the quasi-two-dimensional wave chaotic cavity and the position of the coupling port, while that in (b) shows details of the antenna in cross section.

Fig. 2. PDFs for the (a) real and (b) imaginary parts of the normalized cavity impedance Z_{Norm} for a wave chaotic microwave cavity between 7.2 and 8.4 GHz with h = 7.87 mm and $2a$ = 1.27 mm, for three values of loss in the cavity (open stars: 0, triangles: 2, hexagons: 4 strips of absorber). Also shown are single parameter simultaneous fits for both PDFs with $\tilde{k}^2/Q$ = 0.8, 4.2, 7.6, in order of increasing loss.

Fig. 3. PDFs for the imaginary part of impedance for two values of antenna diameter, $2a$ = 1.27 mm and 0.635 mm, with h = 7.87 mm, 9-9.75 GHz, for (a) cavity reactance and (b) normalized reactance. Differences seen in $Im[Z_{Cav}]$ are largely removed in the normalized PDF.

Fig. 4. Plot of PDF variances for $Re[Z_{Norm}]$ (open) and $Im[Z_{Norm}]$ (closed) for h=7.87 mm and 7.2-8.4 GHz versus fit parameter $\tilde{k}^2/Q$ that simultaneously fits both PDFs. Also shown are similar data for the case h=1.78 mm for $Re[Z_{Norm}]$ (+) and $Im[Z_{Norm}]$ (×) for the 6-7.2, 7.2-8.4, 9-9.75, and 11.25-12 GHz ranges. The solid line is the RCM prediction for the variance dependence on the fit parameter in the time-reversal symmetric wave chaotic case. Inset shows $\tilde{k}^2/Q$ fit values vs. absorber perimeter ratio α for 5 cases (0,1,2,3,4 absorber strips) for the 7.2-8.4GHz range and h = 7.87mm.

Figures:

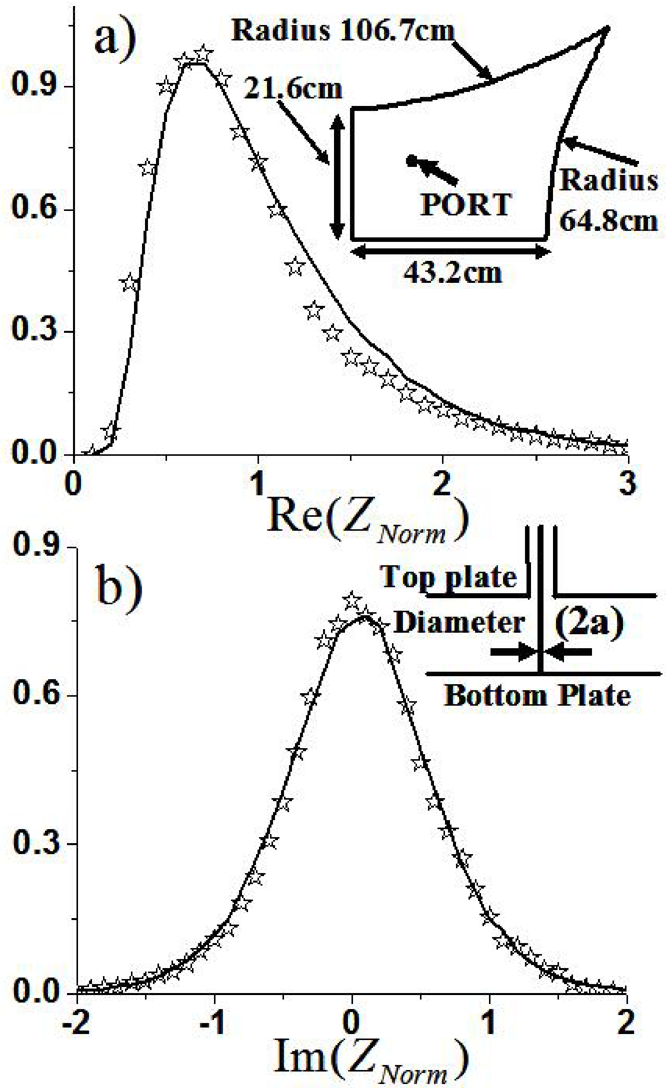

Fig. 1
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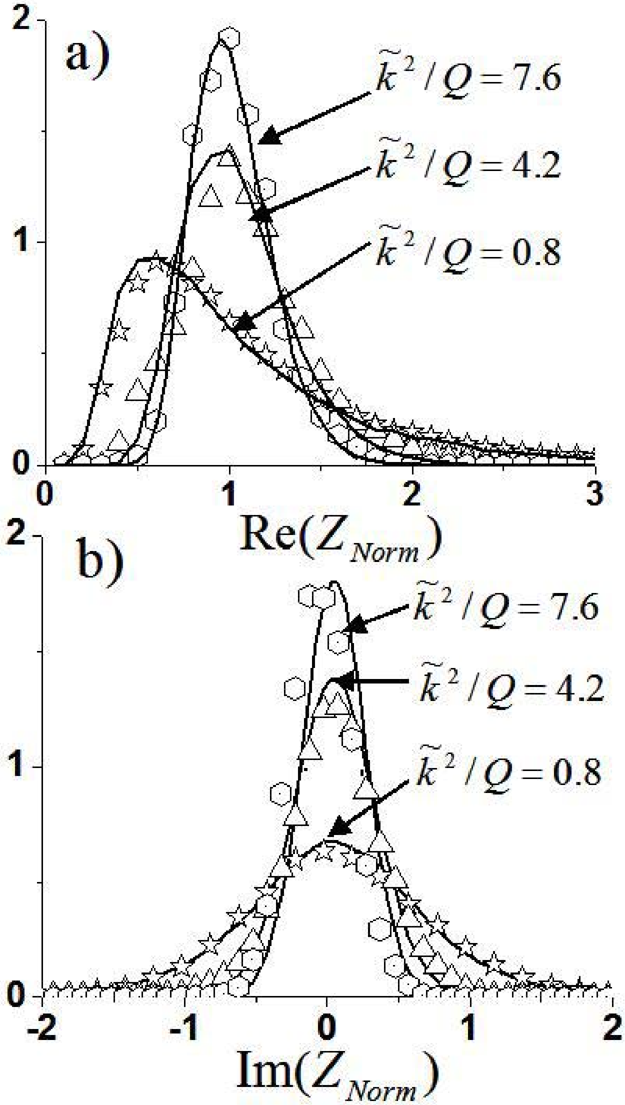

Fig. 2
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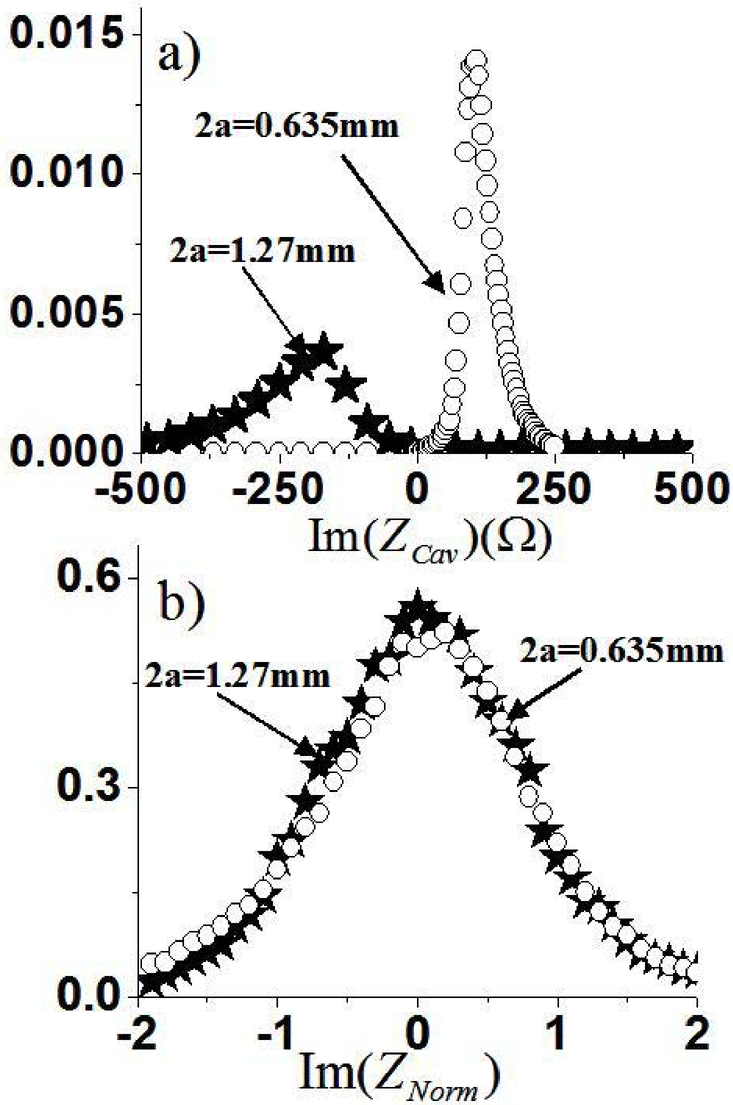

Fig. 3
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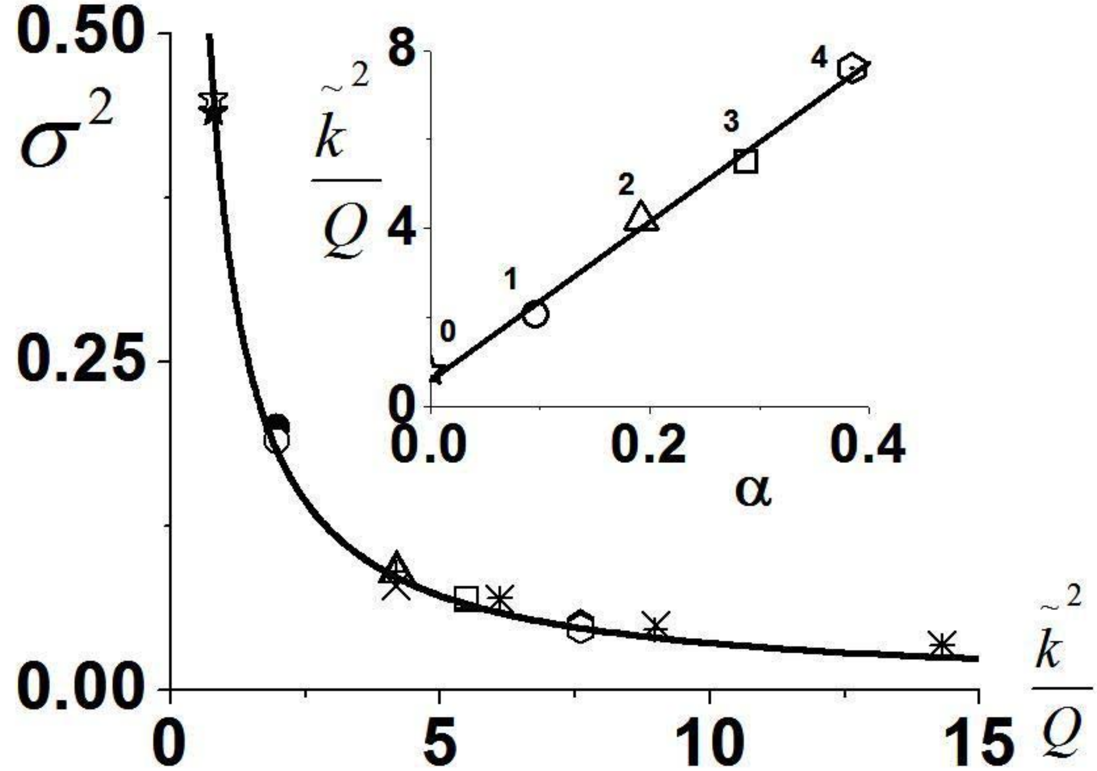

Fig. 4
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