Generating entanglement between quantum dots with different resonant frequencies based on dipole-induced transparency

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We describe a method for generating entanglement between two spatially separated dipoles coupled to optical microcavities. The protocol works even when the dipoles have different resonant frequencies and radiative lifetimes. This method is particularly important for solid-state emitters, such as quantum dots, which suffer from large inhomogeneous broadening. We show that high fidelities can be obtained over a large dipole detuning range without significant loss of efficiency. We analyze the impact of higher-order photon number states and cavity resonance mismatch on the performance of the protocol.

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I. INTRODUCTION

Generation of entanglement between qubits is an important operation for a large variety of applications in quantum-information processing. Such states can be used to realize schemes such as transmission of secret messages via quantum key distribution [1,2] and teleportation of quantum information [3–6]. The exchange of entanglement between two distant parties is also required for implementation of quantum repeaters [7], which use a combination of entanglement swapping and entanglement purification [8] to achieve unconditional secure communication over arbitrarily long distances.

To date, a variety of methods have been proposed for creating entanglement between spatially separated nodes. One of the most common methods is to transmit entangled photons generated by parametric down-conversion [9]. Entanglement protocols for atomic systems have also been proposed [10–14]. Atom entanglement has the advantage that quantum information can be stored for long time periods, which is important for long-distance quantum networking.

Semiconductor-based approaches to quantum-information processing are currently an area of great interest because they offer the potential for a compact and scalable quantum information architecture. Furthermore, solid-state emitters such as semiconductor quantum dots (QDs) can be coupled to ultracompact cavity waveguide systems to form highly integrated quantum systems [15,16]. A major challenge in using solid-state emitters is that they suffer from enormous inhomogeneous broadening, typically caused by emitter size variation and strain fields in the host material. The inhomogeneous broadening makes it difficult to find two emitters with identical emission wavelengths. Protocols to date for generating atom entanglement require the dipoles to emit indistinguishable photons, and are thus difficult to implement in semiconductor systems. In order to implement quantum networking in semiconductors, we need a protocol that works even when the dipoles emit photons that are distinguishable.

In this paper, we describe a protocol for creating entanglement between two dipoles with different radiative properties, such as different emission wavelengths or radiative lifetimes. The proposed protocol uses dipole-induced transparency (DIT) to achieve the desired entanglement, which occurs when a dipole is coupled to an optical cavity [17]. When the coupling is sufficiently strong, the dipole can switch a cavity from being highly transmitting to highly reflecting. The switching contrast is determined by the atomic cooperativity, which is the ratio of the lifetime of the uncoupled emitter to the modified lifetime of the cavity-coupled emitter. Enhancement of spontaneous emission has been observed in semiconductor emitters coupled to a variety of different microcavity architectures [16,18–22]. Modification of cavity reflectivity by coupling a quantum dot to a photonic crystal nanocavity has been recently observed [23,24].

In Sec. II, we describe the basic protocol under idealized assumptions that all fields can be expanded to first order in photon number and that the two cavities have the same resonant frequencies. Section III then considers the effect of higher-order photon number states on the efficiency and fidelity of entanglement. The impact of nonlinear behavior away from the weak excitation limit is also discussed. In Sec. IV we investigate the effect of cavity frequency mismatch on the entanglement. Finally, in Sec. V we perform a precise numerical simulation of the entanglement generation protocol for one specific implementation using the exciton–biexciton cascade of a single indium arsenide (InAs) quantum dot. Numerical results from recent experimental work are used to show that this may be a promising method for achieving entanglement between two QDs.

II. PROTOCOL FOR ENTANGLEMENT GENERATION

The schematic for generating entanglement between two spatially separated dipoles that emit distinguishable photons is shown in Fig. 1. Each qubit consists of a dipole coupled to a double-sided cavity. Each dipole is assumed to have three states: a ground state, a long-lived metastable state, and an excited state, which we refer to as $|g\rangle$, $|m\rangle$, and $|e\rangle$, respectively. The states $|g\rangle$ and $|m\rangle$ represent the two qubit states of the dipole. The transition from the ground state to the excited state for dipole 1 may be detuned by $\delta_1$ from the resonant frequency $\omega_1$ of cavity 1. Similarly, the transition from the ground state to the excited state for dipole 2 may be detuned by $\delta_2$ from the resonant frequency $\omega_2$ of cavity 2. We assume that when the dipole is in state $|g\rangle$ it couples to the cavity
mode via an optical transition to state \( |e\rangle \), while when it is in state \( |m\rangle \) it does not optically couple to the cavity mode. State \( |m\rangle \) may be decoupled from the cavity due to either spectral detuning or selection rules. Although state \( |m\rangle \) is illustrated in the diagram as having an energy level that is in between states \( |g\rangle \) and \( |e\rangle \), this is not required. The only requirement is that when the dipole is in state \( |m\rangle \) it is decoupled from the cavity. This point will be analyzed in more detail when we consider using the exciton-biexciton transitions of an InAs QD to achieve entanglement. The desired level structure described above can be realized in a variety of solid-state material systems. In semiconductor quantum dots one can use the exciton and biexciton transitions [25], as well as the spin-based bright and dark exciton states [26]. In addition, three-level structures can also be achieved using quantum-dot molecules [27], charged quantum dots [28], and impurity-bound excitons [29]. Similar qubit states could also be realized in other materials, such as diamond using neutral and negatively charged nitrogen vacancy defects [30,31].

The decay rates of the two dipoles are given by \( \gamma_1 \) and \( \gamma_2 \), respectively. To characterize the interaction between the dipoles and the cavity modes, we define the operators \( \hat{\sigma}_{1-} \) and \( \hat{\sigma}_{2-} \). They represent the dipole lowering operators for the dipoles in cavities 1 and 2, respectively. It should be noted that \( \hat{\sigma}_{1-} \) and \( \hat{\sigma}_{2-} \) represent the dipole lowering operators for the \( g\to e \) transition. Although state \( |m\rangle \) is decoupled from the cavity, we still define the dipole operators \( \hat{\sigma}_{m1-} \) and \( \hat{\sigma}_{m2-} \) for the dipole in state \( |m\rangle \). These dipole are detuned by \( \delta_{m1} \) and \( \delta_{m2} \) from their respective cavities.

We define \( \hat{a}_m \) and \( \hat{c}_m \) as the input modes, \( \hat{a}_{\text{out}} \) and \( \hat{c}_{\text{out}} \) as the reflected modes, and \( \hat{b}_{\text{out}} \) and \( \hat{d}_{\text{out}} \) as the transmitted modes to the cavities, as illustrated in Fig. 1. The energy decay rate of cavity 1 into the reflected and transmitted modes is given by \( \kappa_1 \) and \( \kappa_{t1} \), respectively. Similarly, the energy decay rates of cavity 2 into the reflected and transmitted modes are given by \( \kappa_2 \) and \( \kappa_{t2} \), respectively. There are also the decay rates \( \kappa_{m1} \) and \( \kappa_{m2} \) into the parasitic leaky modes due to losses such as material absorption and out-of-plane scattering. The fields inside the cavities are represented by the cavity field operators \( \hat{f}_1 \) and \( \hat{f}_2 \).

The protocol works as follows. Both the dipoles are initialized to be in an equal superposition of qubit states \( |g\rangle \) and \( |m\rangle \). This can be achieved by first driving the dipoles into the lowest-energy state by either waiting several radiative lifetimes or optical pumping. The qubit state can then be rotated by either a direct \( \pi/2 \) transition, or a Raman transition [32]. The choice of method depends on the specifics of the dipole and material system. Once the initialization step is complete, the initialized state of the two-dipole system is given by \( |\Omega\rangle = (|gg\rangle + |mm\rangle + |gm\rangle + |mg\rangle)/2 \).

After the initialization of the dipoles, a weak coherent field \( |\alpha\rangle \) with frequency \( \omega \) is inserted at input \( \hat{a}_m \). Simultaneously, another weak coherent field \( |eta\rangle \) that is phase coherent with \( |\alpha\rangle \) (i.e., originates from a common laser source) is injected at \( \hat{c}_m \). These input fields interact with the cavity-dipole system. The interaction between the input field \( \hat{a}_m \) and cavity-dipole system 1 can be characterized by the Heisenberg equations of motion for the cavity field operator \( \hat{f}_1 \) and the dipole lowering operator \( \hat{\sigma}_{1-} \).

\[
\frac{d\hat{f}_1}{dt} = -\left[ i\omega_0 + (\kappa_{t1} + \kappa_{f1} + \kappa_{m1})/2 \right] \hat{f}_1 - \sqrt{\kappa_{t1}} \hat{a}_m - i\sigma_{1-} - i\sigma_{m1-},
\]

\[
\frac{d\hat{\sigma}_{1-}}{dt} = -i(\omega_0 + \delta_{1}) + \gamma_1 \sigma_{1-} + i\sigma_{3} \hat{f}_1,
\]

\[
\frac{d\hat{\sigma}_{m1-}}{dt} = -i(\omega_0 + \delta_{m1}) + \gamma_1 \sigma_{m1-} + i\sigma_{m3} \hat{f}_1.
\] (1)

Similar equations can also be written for the interaction of the input field \( \hat{c}_m \) with cavity-dipole system 2.

The interaction between the input fields and the cavity-dipole systems results in part of the field being transmitted into the modes \( \hat{b}_{\text{out}} \) and \( \hat{d}_{\text{out}} \), while the remainder is reflected into the modes \( \hat{a}_{\text{out}} \) and \( \hat{c}_{\text{out}} \), or absorbed by the QD. The amount of light reflected and transmitted is given by the cavity reflection and transmission coefficients. Our analysis works in the weak excitation limit, where predominantly the quantum dots are populated in the ground state. In this limit, \( \langle \sigma_{3}(t) \rangle \approx -1 \). This also implies that the population inversion for the dipole in state \( |m\rangle \) is close to 0, i.e., \( \langle \sigma_{m3}(t) \rangle \approx 0 \). Using this limit in Eq. (1), we can derive the reflection and transmission coefficients to be [33]

\[
r_{1}(\omega) = \left( \frac{-i\Delta_{1} \omega_1 + \frac{g^2}{-i(\Delta_{1} \omega_1 - \delta_{1}) + \gamma_1} + (\kappa_{t1} - \kappa_{f1} - \kappa_{m1})/2}{-i\Delta_{1} \omega_1 + (\kappa_{t1} + \kappa_{f1} + \kappa_{m1})/2 + \frac{g^2}{-i(\Delta_{1} \omega_1 - \delta_{1}) + \gamma_1}} \right)
\]

\[
r_{2}(\omega) = \left( \frac{-i\Delta_{2} \omega_2 + \frac{g^2}{-i(\Delta_{2} \omega_2 - \delta_{2}) + \gamma_2} + (\kappa_{t2} - \kappa_{f2} - \kappa_{m2})/2}{-i\Delta_{2} \omega_2 + (\kappa_{t2} + \kappa_{f2} + \kappa_{m2})/2 + \frac{g^2}{-i(\Delta_{2} \omega_2 - \delta_{2}) + \gamma_2}} \right).
\]
\[ t_1(\omega) = \frac{\sqrt{\kappa_2 \kappa_1}}{-i \Delta \omega_1 + (\kappa_1 + \kappa_1 + \kappa_1)/2 + \frac{g^2}{-i (\Delta \omega_1 - \delta_1) + \gamma_1}}, \]  

(2)

where \( \Delta \omega_1 = \omega - \omega_1 \). These equations are obtained for cavity-dipole system 1. The reflection and transmission coefficients for dipole 2 are identical to those of dipole 1 in form, and are obtained by substituting the dipole 2 parameters into Eq. (2).

To get a better feel for Eq. (2), it is helpful to first consider the simplified case where \( \Delta \omega_1 = 0 \). Assume first that the dipole is in state \( |g\rangle \) and that the \( g-r \) transition is resonant with the cavity such that \( \delta_1 = 0 \). We see that maximum reflection and minimum transmission occur for the case when \( \kappa_1 = \kappa_1 + \kappa_1 \), called the critical coupling condition. This condition ensures that no light is reflected from the cavity when the incident field is directly on cavity resonance. We represent the decay rate \( \kappa_1 \) at critical coupling as \( \kappa_1 \). Hence, the transmission and reflection coefficients simplify to

\[ t = \frac{1}{1 + C} \]  

and \( r = C/(1 + C) \), where \( C = g^2/\gamma_1 \) is called the atomic cooperativity. If \( C > 1 \), which is the desired operation regime, then \( r = 1 \) and all of the light is reflected. Now suppose the dipole is instead in state \( |m\rangle \) which is detuned from the cavity by \( \delta_2 \). We then have \( t = 1/(1 + CL) \) and \( r = CL/(1 + CL) \), where \( L = \gamma_1/(\gamma_1 + i \delta_2) \) is a Lorentzian function. If we assume that either state \( |m\rangle \) is highly detuned from the resonance of the \( g-r \) transition (\( \delta_2 \gg g^2/\kappa_1 \)) or the transition is very weak due to selection rules (\( C = 0 \)), then \( t = 1 \) and now the light is completely transmitted. Thus, by changing the state of the dipole from \( |g\rangle \) to \( |m\rangle \) we can completely change the reflectivity of the cavity.

In a realistic system we cannot assume that the two dipoles are resonant with their respective cavities, since in general they will have different resonant frequencies. Nor can we assume that the reflection and transmission coefficients will reach their ideal limits because \( \delta_2 \) is not infinitely large and we usually do not have perfect selection rules to cancel out the m-e transition. In this case we define for dipole 1 \( r_1^g \) and \( r_1^m \) as the transmission and reflection coefficients when the dipole is in state \( |g\rangle \), and \( r_1^g \) and \( r_1^m \) when the dipole is in state \( |m\rangle \). We define \( r_2^g, r_2^g, r_2^m, \) and \( r_2^m \) analogously for dipole 2. These coefficients can be calculated by plugging in the appropriate values corresponding to the different transitions of the dipoles. It is important to emphasize that we do not assume that \( \delta, g, \gamma, \) and \( \kappa \) are the same for both dipoles. The protocol we describe works even if all of these parameters are different, which is why it is so useful in semiconductors.

Before continuing, it is worth noting that in much of the literature the atomic cooperativity \( C \) is often interchanged with the Purcell factor, defined as the ratio of the lifetime of the dipole inside the cavity to that of a dipole in bulk or free space, which we denote \( \gamma_{\text{bulk}} \). Although the atomic cooperativity is the correct parameter to use in the strictest sense, it is a very difficult parameter to measure. The Purcell factor in contrast is easier to measure and almost always gives a lower bound on the parameter \( C \) in any realistic system. The reason for this is that \( \gamma_{\text{bulk}} \) is due to both radiative and nonradiative decay. In contrast, the decay rate \( \gamma \) is the decay rate into noncavity modes and is mainly due to nonradiative processes, as radiation into modes other than the cavity mode is highly suppressed. Thus, outside of some atypical cases where the cavity has two modes resonant with the QD, we expect that \( \gamma_{\text{bulk}} > \gamma \). Therefore, in virtually all cases \( C \) can be replaced by the Purcell factor to get a lower bound on the performance of the system.

We first investigate the protocol under the assumptions that the resonant frequencies of both the cavities are the same and that the input fields \( |\alpha\rangle \) and \( |\beta\rangle \) are sufficiently weak that we may expand them to first order in photon number. The initial state of the system (dipoles and fields) is given by

\[ |\Psi_i\rangle = \frac{1}{\sqrt{2}}(|gg\rangle + |mm\rangle + |gm\rangle + |mg\rangle)((a_{\text{in}}^\dagger + b_{\text{in}}^\dagger)). \]

The fields, after interacting with the cavities, are transformed according to cavity reflection and transmission coefficients. The transformation for a photon in the \( \text{in} \) and \( \text{out} \) cavities is called the Purcell factor, defined as the ratio of the lifetime of the cavity to that of a cavity in the bulk. Although the atomic cooperativity \( C \) is often interchanged with the Purcell factor, defined as the efficiency \( \eta \) of the protocol. If we define the field \( |\alpha\rangle \) and \( |\beta\rangle \) weak, the efficiency of the protocol is proportional to the intensity of the field at \( \hat{d}_2 \) and can be derived to be \( |\alpha|^2/4 \). The factor 1/4 appears because 50% of the field is transmitted into the modes \( \hat{b}_{\text{out}}^\dagger \) and \( \hat{d}_{\text{out}}^\dagger \) and another 50% is
lost when the beam splitter splits the photons equally between \( \hat{d}_1 \) and \( \hat{d}_2 \). We see that we can achieve higher efficiencies by increasing the input photon flux rate \( |\alpha|^2 \). However, if we make \( \alpha \) too large we can no longer expand the fields to first order in photon number, and higher-order photon number contributions will become important.

Higher-order photon number contributions are undesirable because they serve as a decoherence mechanism. In the ideal case where only one photon is injected into the system, a detection event at \( \hat{d}_3 \) ensures that there are no other photons in the system which may carry which-path information about the state of the system. Now suppose we consider the second-order process of simultaneously injecting two photons into the input ports \( |\alpha\rangle \) and \( |\beta\rangle \). In the ideal case (both dipoles are on resonance with the cavities), if the state of the two dipoles is \( |gm\rangle \), cavity 1 will reflect its incident photon while cavity 2 will transmit the second photon. The transmitted photon in cavity 2 will always keep track of the fact that dipole 2 was in state \( |m\rangle \), and this information cannot be erased by the beam splitter. Thus, we expect the state to be completely decohered when this happens.

We will now consider not only this specific case, but full expansion of the coherent fields \( \alpha \) and \( \beta \) to all photon numbers to see how the final state of the system is affected. The initial state of the system is given by \( |\Psi_\uparrow\rangle = 1/2(|gg\rangle + |mm\rangle + |gm\rangle + |mg\rangle)|\alpha\rangle |\beta\rangle \). The coherent states \( |\alpha\rangle \) and \( |\beta\rangle \) can also be written as \( |\alpha\rangle = D_1(\alpha)|0\rangle \) and \( |\beta\rangle = D_2(\beta)|0\rangle \). \( D_1 \) and \( D_2 \) are the displacement operators and are given by

\[
D_1(\alpha) = e^{a^\dagger \alpha - \alpha^* a},
\]
\[
D_2(\beta) = e^{b^\dagger \beta - \beta^* b}.
\]

The displacement operator provides a convenient way of writing the coherent states and includes all the higher-order photon number contributions.

The final state of system \( |\Psi_\downarrow\rangle \) can be obtained by applying the cavity and beam splitter transformations to the initial state \( |\Psi_\uparrow\rangle \). After applying the transformations, the final state of the QDs is obtained by tracing out over the photon fields conditioned on a detection event at detector \( \hat{d}_2 \). The state of the dipoles is therefore given by the reduced density matrix

\[
\rho_{\text{dipoles}} = \frac{\text{tr}(\text{fields})\langle M|\langle \Psi_\uparrow|\langle \Psi_\downarrow|M \rangle}{\text{tr}(\text{dipoles+fields})\langle M|\langle \Psi_\uparrow|\langle \Psi_\downarrow|M \rangle}.
\]

The matrix \( M = \sum_{n,m} |n\rangle \langle n| \) is a positive projector that projects the state of the system onto a subspace containing at least one photon in \( \hat{d}_2 \). This projection models the measurement performed by the photon counter, which registers a detection event as long as there is at least one photon in the detection mode.

Since the final state of QDs is mixed, we need a figure of merit to measure how well the QDs are entangled. In this paper we use the fidelity, which is defined as the overlap integral between the desired final state and the actual final state of the system. In our protocol, the desired final state is the maximally entangled Bell state \( |\Psi_\downarrow\rangle \). Thus, the expression for fidelity is \( \langle \Psi_\downarrow|\rho_{\text{dipoles}}|\Psi_\downarrow\rangle \). If the actual final state is the same as the desired final state, we have a perfect entangled state and a fidelity of 1. A fidelity of 0.5 implies that the state of the QDs is a random mixture of \( |gm\rangle \) and \( |mg\rangle \) and completely decohered. An analytical expression for the fidelity can be calculated by evaluating \( \rho_{\text{dipoles}} \) and averaging over the state \( |\Psi_\downarrow\rangle \). We have carried out this calculation, but the expression for the fidelity is messy and the math is involved. The procedure for calculating the fidelity along with the final analytical expression are given in the Appendix. The expression in the Appendix is used for subsequent calculations of fidelity.

We also define the efficiency \( \eta \) as the probability of getting a detection at detector \( \hat{d}_2 \). Mathematically, this is given by the expression \( \eta = \text{tr}(\text{dipoles+fields})\langle M|\langle \Psi_\uparrow|\langle \Psi_\downarrow|M \rangle \). Using the matching condition \( |\alpha|^2 |\beta|^2 = \beta_2^2 \), the expression can be simplified to \( \eta = 0.5(1 - |\alpha|^2 |\beta|^2) \).

For the calculations in this paper, we use parameters that are appropriate for InAs quantum dots coupled to photonic crystal defect cavities. We represent the total decay rate out of each cavity \( \kappa_1 + \kappa_2 + \kappa_l \) and \( \kappa_2 + \kappa_2 + \kappa_2 \) as \( \kappa \) and set it to be equal to 100 GHz. This corresponds to a cavity Q of 3300. We set \( g = 20 \text{ GHz} \) for both the quantum dots. We estimate the dipole decay rate \( \gamma \) within the cavity to be 0.125 GHz, using reported data that measured the lifetime of several quantum dots that were placed inside a photonic crystal cavity, but heavily detuned [34]. Using these values we calculate \( C \) to be 32 and the cavity-dipole systems to be 96.7% reflective on resonance. For the chosen values of \( g \) and \( \kappa \), the cavity-dipole systems are in the weak coupling regime \( g < \kappa/4 \). However, the analysis in this paper is completely general and is equally valid also for the strong coupling regime. In Fig. 2, we plot both fidelity and efficiency as a function of \( |\alpha|^2 \). The fidelity is plotted for four values of \( \delta_1 / \kappa \), ranging from 0 to 1, with \( \delta_2 \) fixed at 0. Note that the efficiency is only a function of \( |\alpha|^2 \), so the plot of efficiency is the same for all values of \( \delta_1 \). From Fig. 2, we see that there is a trade-off between fidelity and efficiency as we increase \( \alpha \). When \( |\alpha|^2 \approx 1 \), the fidelity is close to 1, indicating an ideal entangled state, which is consistent with our predictions in the weak field limit. In the region \( 0.1 < |\alpha|^2 < 1 \), the fidelity quickly drops due to the presence of higher photon number contributions. In the limit \( |\alpha|^2 > 1 \), the fidelity asymptotically approaches 0.5, indicating that the higher photon number contributions have completely decohered the state.
When \( |\alpha|^2 \ll 1 \), the fidelity curves for different values of \( \delta_r \) nearly overlap. The fidelity stays close to 1 in this region. However, in the region \( 0.1 < |\alpha|^2 < 1 \), the fidelity curves for different values of \( \delta_r \) separate out. There is a drop in fidelity with increase in dipole detuning from \( \delta_r = 0 \) to \( \delta_r = \kappa \). Also, efficiency is a function of \( |\alpha|^2 \) and does not change with \( \delta_r \). This implies that efficiency decreases with increase in dipole detuning for a constant efficiency.

In Fig. 2, we also plot a line of constant fidelity of 0.85. Note that for every value of \( \delta_r \), there is a unique point on the plot corresponding to a fidelity of 0.85. As \( \delta_r \) increases, this point shifts to lower values of \( |\alpha|^2 \). Since efficiency is a function of \( |\alpha|^2 \), this in turn implies a decrease in efficiency. Thus, it is important to consider how the efficiency of the protocol changes for a fixed value of fidelity.

To investigate this, we plot in Fig. 3 the efficiency as a function of \( \delta_l / \kappa \) for several values of \( \delta_r \) for a constant fidelity of 0.85. We see that even though there is a loss of efficiency, the change is gradual and there is only a 50% reduction over a cavity linewidth. Also, we would expect that if we added another detuning \( \delta_l \), efficiency would decrease. However, this does not happen. From Fig. 3 we see that the effect of \( \delta_r \) is to shift the efficiency curves by the detuning \( \delta_r \) without altering the shape. So the protocol can be used to obtain high efficiencies over a wide range of dipole detunings.

**IV. VALIDITY OF WEAK EXCITATION LIMIT**

In the protocol we describe, dipole detuning is compensated by adjusting the amplitude and phase of the input coherent fields until the matching condition \( a_1^\dagger = \beta \sigma_+^\dagger \) is satisfied. The more detuning we have, the larger the amplitude required by the coherent field in order to achieve the desired efficiency. It is possible that at some point, the amplitudes required by the coherent fields will be so large that the g-e transition of the QDs will be saturated, leading to an optical nonlinearity and linewidth broadening [35]. Because of this, the cavity reflection and transmission equations will depend on the pump power, and Eq. (2) needs to be modified accordingly. However, our protocol is intended to work in the linear regime wherein the QDs are unsaturated. This is possible only if the amplitude of the input fields is within a certain limit called the weak excitation limit. The weak excitation limit is defined as \( \langle \sigma_e(t) \rangle \ll 1 \), which is equivalent to the statement \( \langle \sigma_e(t) \rangle \ll 1 \), and is necessary for Eq. (2) to be valid. This condition puts a constraint on the operation of the protocol.

In order to investigate the implication of the weak excitation constraint, we start with the Heisenberg equation of motion for the cavity field operator \( \hat{f}_1 \) and the dipole lowering operator \( \hat{\sigma}_- \) given in Eq. (1). We will consider cavity-dipole system 1. Similar equations are also applicable for cavity-dipole system 2.

Eliminating \( b \) from Eq. (1), we have

\[
\left( \frac{\kappa}{2} (i \delta_r + \gamma) - g^2 \right) \hat{\sigma}_- = -ig \sqrt{\kappa} a_{in}^\dagger.
\]

Using the fact the cooperativity index \( C \) is \( g^2 / \gamma \kappa > 1 \), the equation can be further simplified and multiplied with its conjugate to obtain

\[
\langle \sigma_e \sigma_e \rangle = \frac{g^2 \kappa}{(g^2 + \delta_l \kappa / 4)} \langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle.
\]

The parameter \( \langle \sigma_e \sigma_e \rangle \) represents the probability of the QD being in the excited state. In the weak excitation limit, \( \langle \sigma_e \sigma_e \rangle \ll 1 \). We also identify \( \langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle \) as the total flux of photons in the input field \( |\alpha| \). Using this in Eq. (8), the weak excitation constraint thus puts a limit on \( |\alpha|^2 \) given by

\[
\frac{|\alpha|^2}{\tau_p} \ll \frac{g^2}{\kappa} + \frac{\kappa \delta_l^2}{g^2},
\]

where \( \tau_p \) is the pulse width of the laser.

From Eq. (9), we see that when there is no detuning \( \delta_l \), the flux of photons in the input field \( |\alpha| \) should be less than the modified lifetime of the QD within the cavity \( g^2 / \kappa \). This is understandable because, if the first photon excites the QD and the second photon comes in before the QD has decayed, we will no longer be in the weak excitation limit. However, if the QD is off resonant from the cavity with detuning \( \delta_l \), not all the light that comes in couples to the QD. Therefore, we will be able to pump the QDs with much more power before we exceed the weak excitation limit. This is given by the detuning-dependent term \( \kappa \delta_l^2 / g^2 \) in Eq. (9).

Equation (9) conveys more than the weak excitation limit of \( |\alpha| \). If we apply the matching condition \( a_1^\dagger = \beta \sigma_+^\dagger \) in Eq. (9), we obtain a limit on the flux of photons in the input field \( |\beta| \) given by

\[
\frac{|\beta|^2}{\tau_p} \ll \frac{g^2}{\kappa} + \frac{\kappa \delta_l^2}{g^2}.
\]

We recognize this as the weak excitation limit equation for the field \( |\beta| \) which we would have obtained had we used the Heisenberg equations of motion for cavity-dipole system 2. This implies that if we pick \( |\alpha| \) such that it satisfies the weak excitation limit of cavity-dipole system 1, the matching condition automatically ensures that the flux of photons in \( |\beta| \) is within the weak excitation limit of cavity-dipole 2.

Note that, by making \( \tau_p \) sufficiently long, we can always ensure that the system is in the weak excitation limit and that nonlinearities do not contribute. However, because we are using longer pulses the entanglement rate is reduced. The
rate of entanglement generation is proportional to the rate at which the cavity reflects photons, given by $R = |\alpha r^d|^2/\tau_p$. Using the upper limit on $|\alpha|^2/\tau_p$ from Eq. (9) and cavity reflectivity $r^d$ from Eq. (2), we get

$$R \ll \frac{g^2}{\kappa}. \quad (11)$$

The above equation implies that the system will remain in the linear weak excitation limit provided that the rate of reflected photons is less than one photon per modified lifetime of the dipole. Note that this result is true regardless of the detunings, and is therefore valid in all cases.

It is instructive to compare the limits on the entanglement rate imposed by nonlinearities to the limits imposed by which-path information given in Sec. III. The analysis of higher-order photon numbers in the previous section showed that reflected photons $|\alpha r^d|^2 \ll 1$ to have a high-fidelity entangled state between the QDs. In contrast, the analysis of the weak excitation limit in this section puts an upper bound on the rate of the input photons in $|\alpha|$ and $|\beta|$ given by $|\alpha|^2/\tau_p \ll g^2/\kappa$. Thus, the two analyses are fundamentally different in that one limits the total number of input photons and the other limits the rate of incoming photons. Although one might expect the nonlinear limit analyzed in this section to be important, it turns out that the analysis of Sec. III is more restrictive, and is therefore the important limit to consider. To understand why, we first note that nonlinearities can always be suppressed by increasing the pulse duration $\tau_p$. No matter how many photons we inject into the system, if we make the pulses sufficiently long we will always be in the weak excitation limit. In contrast, which-path information does not depend on pulse duration, and therefore cannot be suppressed. Furthermore, in order to stay in the monochromatic limit (i.e., to use the single-frequency approximation) it has been shown in previous work that the pulse duration must be longer than the modified spontaneous emission lifetime of the dipole [17]. If we combine this with the results of Sec. III, which state that the number of reflected photons $|\alpha r^d|^2 \ll 1$, these two conditions already constrain us to work in the regime where $|\alpha|^2/\tau_p \ll g^2/\kappa$. Thus, we expect the entanglement to decohere due to which-path information before the nonlinear behavior in this section is observed. For this reason deviation from weak excitation does not pose any additional restrictions to the protocol that were not already present in the linear scattering regime.

V. EFFECTS OF CAVITY DETUNING

In previous sections, we considered the idealized case where both the cavities had identical resonant frequencies. However, in realistic systems, this will not be the case. Fabrication imperfections may lead to slightly different resonances for the two cavities. Clearly, if even a small amount of mismatch between the cavities were to result in no entanglement, the usefulness of our protocol would be questionable. Thus, it is important to consider how sensitive the protocol is to cavity resonance mismatch.

Now let us consider the case where the two cavities do not have the same resonant frequency. The analysis of the protocol in the presence of cavity detuning becomes involved for two reasons. First, it is no longer clear which frequency we should use for the coherent fields $|\alpha|$ and $|\beta|$. We do not know whether to place it on resonance with one of the cavities or somewhere in between. This can depend on both the cavity separation $\Delta \omega_1$ and dipole detunings $\delta_1$ and $\delta_2$.

Second, the matching condition used in the previous section $\alpha^d = \beta r^s_1$, is not guaranteed to be optimal. If a detection event is observed in detector $d_2$, then the state of the two QDs is

$$|\Psi_{j}\rangle_{\text{dipoles}} = \frac{1}{N}[(\alpha^d - \beta^s_1)|gg\rangle + (\alpha^d - \beta^s_2)|mm\rangle$$

$$+ \alpha^s_1 |mg\rangle - \beta^s_2 |gm\rangle], \quad (12)$$

where $N^2 = |\alpha^s_1 - \beta^s_2|^2 + |\alpha^s_1|^2 + |\beta^s_2|^2$. The matching condition $\alpha^d = \beta r^s_1$ ensures that we do not have any detection at $d_2$ if both the dipoles are in the state $|g\rangle$. However, the field amplitude at $d_2$ if both the dipoles are in the state $|m\rangle$, i.e., $(\alpha^s_1 - \beta^s_2)$ is not compensated. This results in imperfect destructive interference at detector $d_2$. Thus, there is a small probability of detection at $d_2$ when both the dipoles are in state $|m\rangle$. This causes a loss of fidelity. In order to obtain the state that comes closest to the desired entangled state, we must optimize the fidelity with respect to $\omega_1$, $\alpha$, and $\beta$.

For calculating the effects of cavity detuning, we choose the frequency midway between the two cavity frequencies as the reference frequency $\Delta_{\text{ref}}$. Based on this reference frequency, $\omega_1 = -\Delta \omega_1/2$ and $\omega_2 = \Delta \omega_1/2$. Also, it will be easier if we define the dipole detunings in terms of the reference frequency rather than the cavity frequencies. We define $\Delta_1 = \delta_1 + \omega_1$ and $\Delta_2 = \delta_2 + \omega_2$, which are the dipole detunings of dipoles 1 and 2 with respect to the reference frequency located midway between the two cavities. These definitions ensure that when increasing the cavity separation $\Delta \omega_1$ we do not affect the QDs. This is important because we can obtain information about the effects of cavity detuning alone by making these definitions.

Figure 4 plots the dependence of fidelity on the laser frequency for several different values of $\Delta_1$. The cavity separation $\Delta \omega_1 = \omega_2 - \omega_1$ is set to 50 GHz, and $\Delta_2 = 0.25 \kappa$. The fig-
ure is optimized over the real and imaginary parts of $\alpha/\beta$. The value of the maximum fidelity for the three curves occurs at three different frequencies. The frequency at which we get maximum fidelity is the optimal frequency $\omega$. The fidelity at that frequency is the maximum fidelity that can be obtained for that particular configuration of $\Delta\omega_1$, $\Delta_1$, and $\Delta_2$.

In Fig. 5, we plot the optimized fidelity as a function of cavity detuning $\Delta\omega_2$ for different values of $\Delta_1$ with $\Delta_2=0$. When $\Delta\omega_2=0$, which represents the case when there is no cavity detuning, the fidelity is 1. As the two cavities move apart, the spectra of the two cavities no longer overlap. Thus, there is a small probability of photon detection at $\text{d}_3$ when the dipoles are in the state $|mm\rangle$. This results in a loss of fidelity. Surprisingly, however, the fidelity does not continue to decrease, but instead increases back to 1 at some value of $\Delta\omega_2$.

As we keep increasing $\Delta\omega_2$ further, for a certain value of the laser frequency $\omega$, both $r_1$ and $r_2$ are 0. If a detection event is observed in detector $\text{d}_2$, then the state of the two QDs collapses to

$$|\Psi\rangle_{\text{dipoles}} = \frac{1}{N} \left[ (\alpha r_1^m - \beta r_2^m)|mm\rangle + \alpha r_1^g|gm\rangle + \beta r_2^g|gm\rangle \right],$$

(13)

where $N^2 = |\alpha r_1^m - \beta r_2^m|^2 + |\alpha r_1^g|^2 + |\beta r_2^g|^2$. In this special case, there is a second matching condition, given by $\alpha r_1^g = \beta r_2^g$, that again projects the two dipoles onto $|\Psi\rangle = (|gm\rangle - |gm\rangle)/\sqrt{2}$. It is this second matching condition that results in the fidelity of 1 at the second peak. Our optimization algorithm naturally detects these two optimal regions, and gives us the best performance in the intermediate regime. Thus, given any set of operating conditions we have the ability to determine the best set of amplitudes and input frequencies. We note that in many cases fidelities exceeding 0.95 can be achieved even with a 60 GHz detuning, which is more than half a cavity linewidth. The fabrication of cavities with resonance frequencies that are repeatable within a linewidth is well within current technological capabilities.

We can also consider what happens when we have both cavity detuning and dipole detuning. In Fig. 6, we plot the optimized fidelity as a function of cavity detuning $\Delta\omega_2$ and dipole detuning $\Delta_1$. A maximum fidelity of 1 is obtained when $\Delta\omega_2=0$, which represents the case when the two cavi-}

![FIG. 5. (Color online) Optimized fidelity as a function of $\Delta\omega_2$ for different values of $\Delta_1$. $\Delta_2=0$.](image5.png)

![FIG. 6. (Color online) Fidelity as a function of cavity separations $\Delta\omega_2$ and dipole detuning $\Delta_1$.](image6.png)

![FIG. 7. (Color online) QD as a three-level system.](image7.png)
ground, X, and XX states. We are free to assign these three states as \(|g\), \(|m\), and \(|e\) in a variety of different combinations. In fact, there are several ways to assign these levels, but probably the most convenient approach is given in the inset of Fig. 7. In the figure we have identified the ground state of the QD as state \(|m\), the single-exciton state as state \(|g\), and the biexciton state as state \(|e\). This choice of the level configuration has a number of advantages. First, single-qubit operations between \(|g\) and \(|m\) can be directly applied by pulses resonant with the single-exciton transition. Second, by placing the biexciton transition on resonance with the cavity, we can enhance the exciton to biexciton transition to get DIT, while at the same time suppressing the single-exciton lifetime in order to increase the coherence time of the qubit. This is illustrated in Fig. 7.

We assume the biexciton transition to be on resonance with the cavity frequency. This is indicated in Fig. 7 where the XX transition is in the middle of the cavity spectrum. The X transition line is detuned from the cavity by \(\delta_0\). The vacuum Rabi frequencies of X and XX transitions are given by \(g_X\) and \(g_{XX}\) respectively. Similarly, the decay rates of the two transitions are given by \(\gamma_X\) and \(\gamma_{XX}\), respectively.

In the exciton-biexciton scheme the degree of cavity enhancement directly impacts our ability to create an entangled state. This is because both the exciton and biexciton are strongly radiative states, and the only way to enhance one while suppressing the other is to use cavity lifetime modification. In other qubit implementations, such as dark state excitons [26], this is not as much of a problem because selection rules make the dark exciton long lived regardless of cavity. To quantitatively address this issue, we first calculate the coherence time of the exciton state which is given by solving for the decay rate of \(\sigma\) in Eq. (1). The coherence time of the qubit is given by

\[
\Gamma_X = \frac{g_{XX}^2 \kappa}{\delta_X^2 + \kappa^2} + \gamma_X + \frac{1}{T_2},
\]

where we have added the dipole dephasing rate \(1/T_2\) to the decay rate. From the above equation, one can see that increasing \(\delta_X\) decreases the decoherence rate until it finally saturates at a minimum value of \(\gamma_X + 1/T_2\). At this point, increasing the detuning of the exciton will not help as we are limited by nonradiative and dephasing processes.

The coherence time of the dipole should be compared to an appropriate time scale in order to determine if entanglement can be generated. Although there are a number of different factors that should be considered in this comparison, the minimum requirement for generating entanglement is that the duration of the entangling pulse should be shorter than the coherence time of the qubit. If this is not the case, the qubit will begin to decohere before the entangling pulse has finished interacting with the cavity-dipole system, and there is no hope of generating high-fidelity entanglement. In previous work in Ref. [17], it has been shown that, when the pulse is resonant with the dipole, it must be much longer than the modified spontaneous emission lifetime of the dipole in order to be monochromatic. Thus, in the worst case when the dipole is resonant with the cavity we need \(1/\tau_{g} \leq g_{XX}/\kappa\). We thus argue that an important figure of merit is the ratio of the coherence time of the qubit to the entanglement pulse width, given by

\[
N_{\text{ent}} = \frac{g_{XX}^2}{\kappa \Gamma_X}.
\]

This ratio determines the maximum number of entanglements that can be performed before the system decoheres. If \(N_{\text{ent}} > 1\), there is enough time for the pulses to finish their interaction with the QDs before the system has decohered. Otherwise, the QDs will start to decohere before the pulses have finished their interaction and high-fidelity entanglement will be impossible.

For calculations, we choose experimental values taken from the paper of Hennessy et al. [37], which investigates the coupling of an indium arsenide (InAs) quantum dot coupled to a photonic crystal cavity patterned in gallium arsenide (GaAs) by electron beam lithography. This experimental work reports \(g=20\,\text{GHz}\) and cavity linewidth of 25 GHz, which correspond to a Q of 13 300. However, the cavity linewidth is the bare cavity Q which corresponds to the decay into the leaky modes. In order to achieve critical coupling with the cavity, we need another in-plane mode with a decay rate equal to the bare cavity decay rate. This mode can be implemented in a photonic crystal as a waveguide coupled to the cavity. Thus, the total decay rate of the cavity is double the bare cavity decay rate. Hence, we use \(\kappa=50\,\text{GHz}\) in our calculations. We use \(g_X=g_{XX}=g\). For values of \(T_2\) we use 2 ns, which are appropriate values for InAs QDs [38]. For these values, \(g_{XX}^2/\kappa = 8\,\text{GHz}\), \(\Gamma_X = 0.93\,\text{GHz}\), and \(N_{\text{ent}} = 8.6\). The fact that \(N_{\text{ent}} \geq 1\) ensures that we can complete an entanglement operation well before the QDs have decohered.

For a cavity linewidth of 50 GHz, the exciton line lies outside the cavity spectrum (\(\delta_0=250\,\text{GHz}\)). However, the exciton line still couples to the cavity and we cannot ignore the presence of the extra transition coupled to the cavity. So we cannot substitute for \(g\) by 0 in Eq. (2) in order to obtain the cavity reflection and transmission equations when the QD is in state \(|m\). We need to use the vacuum Rabi frequency as the value for \(g\) to obtain the values of \(r_1^m\), \(r_2^m\), \(r_1^e\), and \(r_2^e\). The changes in the transmission and reflection coefficients will modify the final state of the QDs and hence the fidelity of the system.

In general, we cannot assume that the XX transition is not detuned from the cavity spectrum. In order to see how robust the biexciton-exciton protocol is to dipole detunings, we define the detunings of the XX transition lines from their cavities as \(\delta_{XX1}\) and \(\delta_{XX2}\). In Fig. 8 we plot the dependence of fidelity on dipole detunings \(\delta_{XX1}\) and \(\delta_{XX2}\) for the above case. For both \(\delta_{XX1}=0\) and \(\delta_{XX2}=0\), the fidelity is 1 as expected. When we increase \(\delta_{XX1}\) and \(\delta_{XX2}\), the transmission and reflection coefficients are modified due to the coupling of the X transition to the cavity. This lowers the fidelity of the output state. The drop in fidelity is gradual, and for a cavity linewidth separation of the dipoles from the cavity resonance (50 GHz), the fidelity drops to only 0.96. As we further increase the detunings to 100 GHz, the fidelity drops to 0.85.
Thus, even for large detunings between the cavities and the
dipoles, reasonably high-fidelity (0.85) states of the QDs can
be obtained. Thus, the exciton-biexciton scheme can be used
to create entanglement between QDs even if the exciton line
couples to the cavity. The performance of the protocol can be
further improved by fabricating cavities with high quality
factors.

VII. CONCLUSIONS

In conclusion, we have shown that one can achieve high-
fidelity entangled states between two dipoles, even when
their emission frequencies are different. The method is robust
to dipole and cavity frequency mismatch. The efficiency loss
for a cavity linewidth change in dipole detuning is about
50% for a constant fidelity. Therefore, relatively high fidelity
can be obtained over a large range of dipole detunings with-
out significant loss of efficiency. The development of proto-
cols that are robust to these imperfections is extremely im-
portant for semiconductor-based implementations of quantum
networks.

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APPENDIX

The input coherent fields are $|\alpha\rangle$ and $|\beta\rangle$. Both the dipoles
are initialized in a superposition of states $|g\rangle$ and $|m\rangle$. Thus,
the initial state of the system is

$$\Psi_1 = |\alpha\rangle|\beta\rangle(|g\rangle + |m\rangle)(|g\rangle + |m\rangle)/2. \quad (A1)$$

The coherent states can be replaced by their corresponding
displacement operators to account for all order of photon
numbers. Thus,

$$\Psi = \frac{1}{\sqrt{2}} e^{(\alpha a_0^\dagger - \alpha^* a_0)} e^{(\beta c_0^\dagger - \beta^* c_0)} |0\rangle_{c_{in}} \times (|g\rangle + |m\rangle + |mg\rangle + |mm\rangle). \quad (A2)$$

The input fields after interactions with the cavity-dipole
systems are transformed according to Eq. (2). Thus, when the
dipoles are in state $|gg\rangle$,

$$a_{in}^\dagger \rightarrow r_1^g a_{out}^\dagger + r_2^g b_{out}^\dagger,$$

$$c_{in}^\dagger \rightarrow r_2^c c_{out}^\dagger + r_3^c d_{out}^\dagger. \quad (A3)$$

Similar transformation equations apply when the dipoles
are in the states $|gm\rangle$, $|mg\rangle$, and $|nm\rangle$. The reflected field
from the two cavities is mixed on a 50:50 beam splitter that
applies the transformation

$$a_{out}^\dagger \rightarrow (d_1 + d_2)/\sqrt{2},$$

$$c_{out}^\dagger \rightarrow (d_1 - d_2)/\sqrt{2}. \quad (A4)$$

Applying the cavity and beam splitter transformations on
the initial state $|\Psi_1\rangle$ we get

$$|\Psi_{gg}\rangle = D(\frac{\alpha r_1^g + \beta r_2^g}{\sqrt{2}}) D(\frac{\alpha r_2^c - \beta r_3^c}{\sqrt{2}}) D(\alpha r_1^c) D(\beta r_2^c) \times (|0\rangle_{d_1} |d_2^c} |d_{out}^c} |gg\rangle). \quad (A5)$$

$|\Psi_{gg}\rangle$ is the state of the output modes for the dipoles in state
$|gg\rangle$. This state can be split up into the modes of detector $d_1$
and $b_{out}$ and $d_{out}$, and detector $d_2$. Thus,

$$|\Psi_{gg}\rangle = |\mu_{gg}\rangle |gg\rangle,$$

$$|\mu_{gg}\rangle = D(\frac{\alpha r_1^c - \beta r_2^c}{\sqrt{2}}) D(\alpha r_1^c) D(\beta r_2^c) |0\rangle_{d_1},$$

$$|\Psi_{gg}\rangle = |\Psi_{gg}\rangle + |\Psi_{gm}\rangle + |\Psi_{mg}\rangle + |\Psi_{mm}\rangle. \quad (A7)$$

These states $|\Psi_{gm}\rangle$, $|\Psi_{mg}\rangle$, and $|\Psi_{mm}\rangle$ can be further decom-
posed on similar lines to Eq. (A6) to obtain the field amplitudes
$|\mu_{gm}\rangle$, $|\mu_{mg}\rangle$, and $|\mu_{mm}\rangle$, $|\mu_{mm}\rangle$, respectively.

We define the projection matrix $M$ as $\sum_{n\in\infty}^\infty |n\rangle_{d_2} \langle n|$. $M$ can also be written as

$$M = \sum_{n=0}^\infty |n\rangle_{d_2} \langle n| - |0\rangle_{d_2} \langle 0| = I - |0\rangle_{d_2} \langle 0|, \quad (A8)$$

$$\rho_{\text{dipoles}} = \frac{\text{tr}(|\rho\rangle_{\text{fields}} (\langle M|\Psi_f\rangle (\Psi_f\rangle M))}{\text{tr}(|\rho\rangle_{\text{dipoles+fields}} (\langle M|\Psi_f\rangle (\Psi_f\rangle M)). \quad (A9)$$
The denominator is the probability of getting a detection at detector $\hat{d}_2$. We identify this as the efficiency $\eta$.

$$F = \frac{1}{\eta} \left( \langle \Psi_\perp | \rho_{\text{dipoles}} | \Psi_\perp \rangle - \frac{\text{tr}(\text{fields}) (\langle \Psi_\perp | \Psi_\uparrow \rangle \langle \Psi_\uparrow | M \rangle) - \text{tr}(\text{fields}) (\langle 0 | \Psi_\uparrow \rangle \langle \Psi_\uparrow | 0 \rangle_{d_2})}{2} \right)$$

$$= F_1 - F_2,$$  \hspace{1cm} (A11)

The individual terms can be evaluated to give

\begin{align*}
F_1 &= \frac{1}{4} \left( \frac{\langle \Psi_{\text{gml}} | \Psi_{\text{mgl}} \rangle}{2} - \frac{\langle \Psi_{\text{mlg}} | \Psi_{\text{mgl}} \rangle}{2} \right), \\
F_2 &= \frac{1}{8} e^{-|\mu_{\text{mlg}}|^2} + e^{-|\mu_{\text{mlg}}|^2} - e^{-|\mu_{\text{mlg}}|^2} |\mu_{\text{mlg}}|^2 \langle \Psi_{\text{mgl}} | \Psi_{\text{mgl}} \rangle, \\
\eta &= \frac{1}{4} \left( e^{-\mu_{\text{dip}}^2} + e^{-\mu_{\text{dip}}^2} + e^{-\mu_{\text{dip}}^2} + e^{-\mu_{\text{dip}}^2} \right). \hspace{1cm} (A12)
\end{align*}

Thus, the complete expression for fidelity and efficiency can be obtained.