Markov partitions of maps on 2d tori and their relationship to mixing

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Image borrowed from the cover of *Introduction to the Modern Theory of Dynamical Systems* by A. Katok and B. Hasselblatt

Small scale flows are deterministic – chaos can be employed to improve mixing effeciency

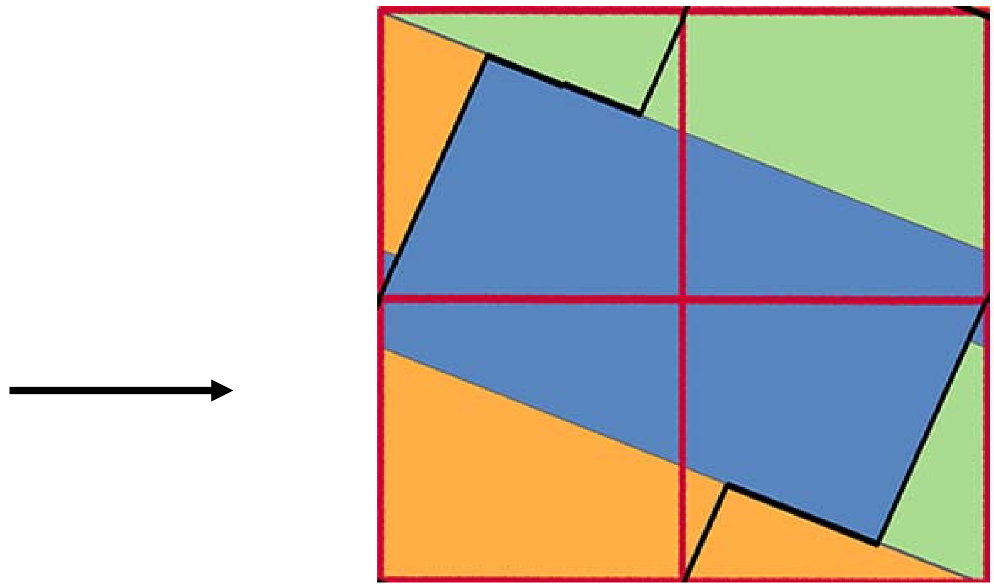
Microfluidics has many developing applications, but mixing on the small scale is difficult to achieve in the usual way.

"Viscous effects dominate at small scales and viscosity-dominated flows are deterministic. Inducing turbulence – making the fluid random to improve mixing – is typically impractical."

– J.M. Ottino and S. Wiggins in *Science*, 24 July 2004

At alternative is chaotic mixing, which uses deterministic processes that are known to be chaotic as a means of mixing things up. It is important to know the dynamics of the processes you are using. Tools from dynamical systems can clearly be applied here.

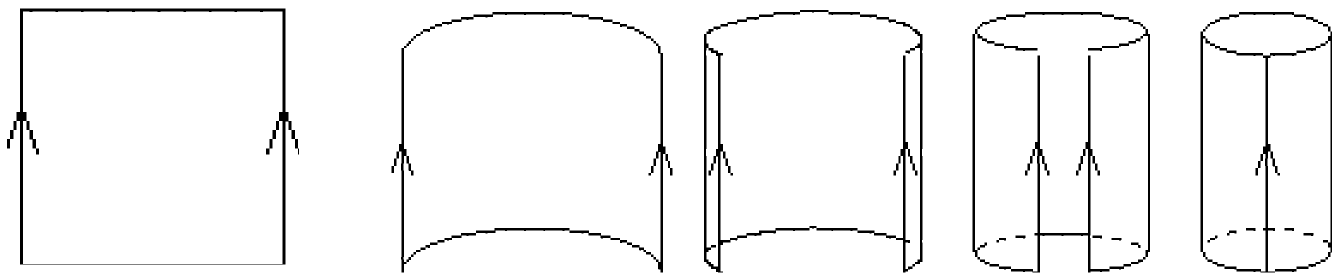
Some mixing mechanisms can be reduced to linear maps on surfaces



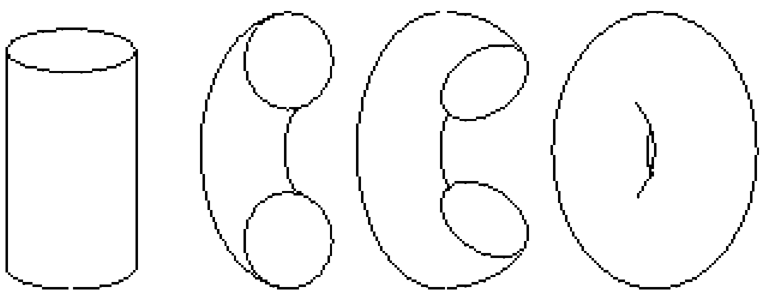
For instance, J.T. Halbert and J.A. Yorke have shown that a linear map on the set at right, viewed as a sphere, describes the motion of the taffy machine at left. The model can then be used to evaluate how efficiently the process mixes.

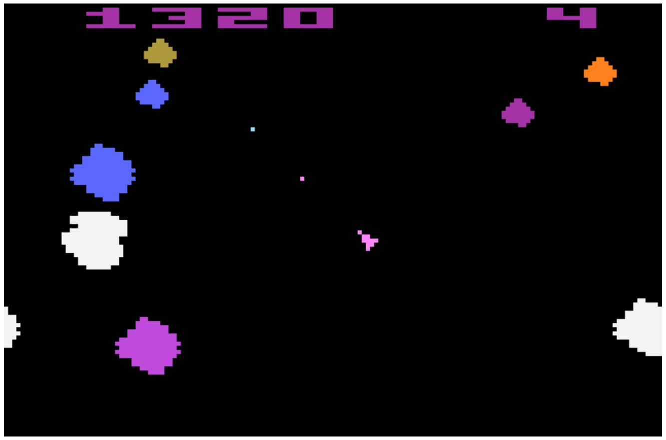
Taffy image from Duke Univesity Dept. of Physics. Right image by J.T. Halbert and J.A. Yorke.

Picturing the unit square as a torus:



Each edge point is identified with the corresponding point on the opposite edge. The classic arcade game Asteroids is set on a such a space.





Torus images borrowed from the home page of Professor James Schombert, Dept. of Physics, University of Oregon.

Our Markov partitioning method

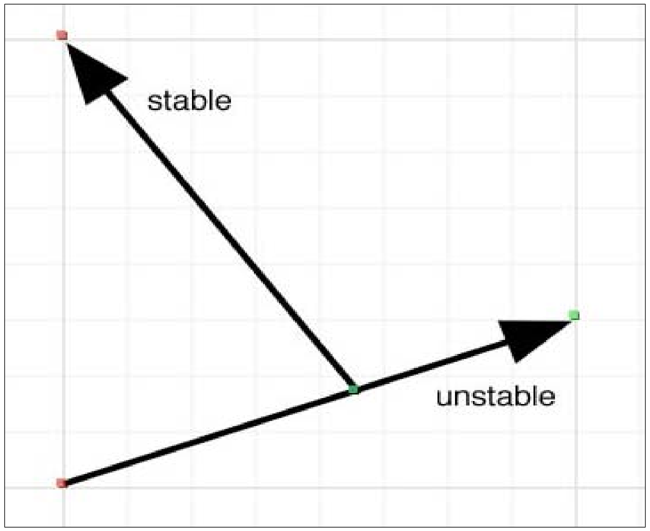
The typical prescription for finding a Markov partition is to draw eigenlines from a fixed point until they intersect, forming parallelograms that may or may not be a Markov partitioned (and thus must be checked). We propose a systematic approach that would ideally work for any hyperbolic map on the torus, but will be shown to have some limitations.

The main idea of our approach is that whenever stable and unstable manifolds intersect, the **longer line stops and the shorter line continues**.

For partitions on the torus, we found that either three eigenlines or four must be drawn.

Finding the Markov partition for the map $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

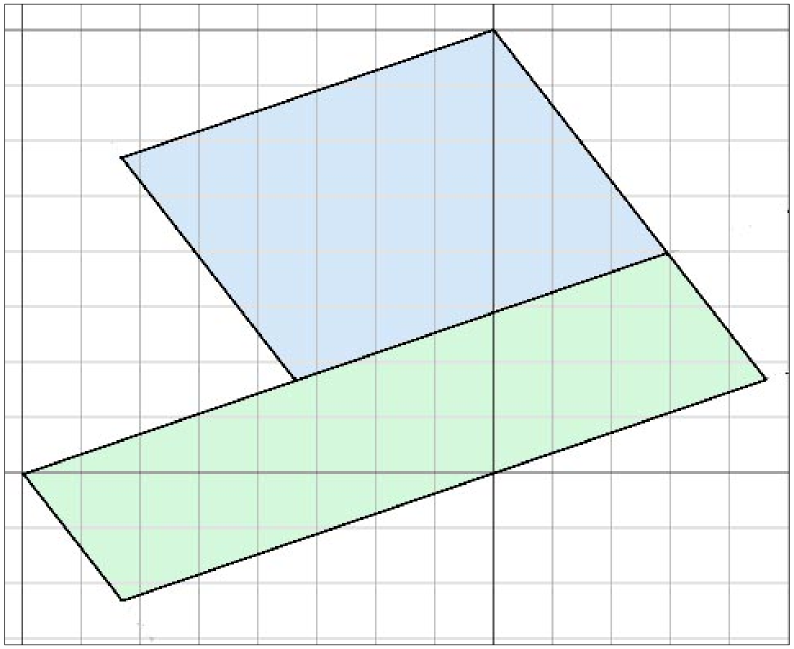
This map has eigenvectors $\begin{pmatrix} 1+\sqrt{3} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1-\sqrt{3} \\ 1 \end{pmatrix}$



Then the origin, which is a fixed point, has stable and unstable manifolds in the directions of these vectors.

Finding the Markov partition for the map $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

This is the proposed Markov partition, found using our method. Three eigenlines are used to produce a two box partition. Each line starts at one of the three corners of the torus.



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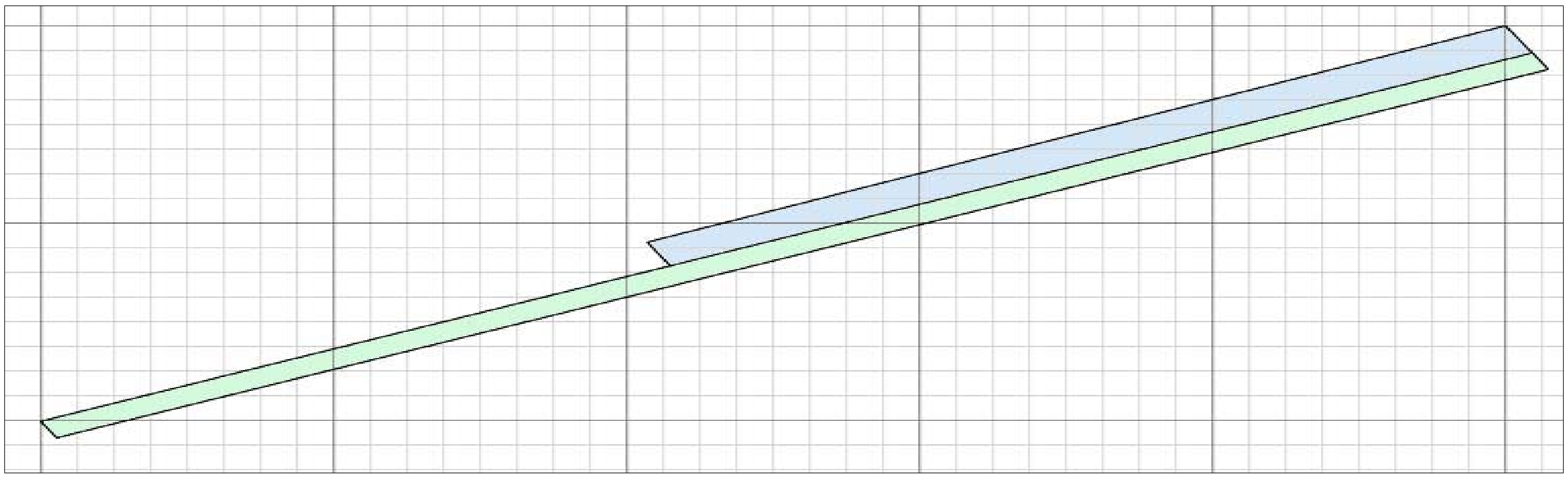
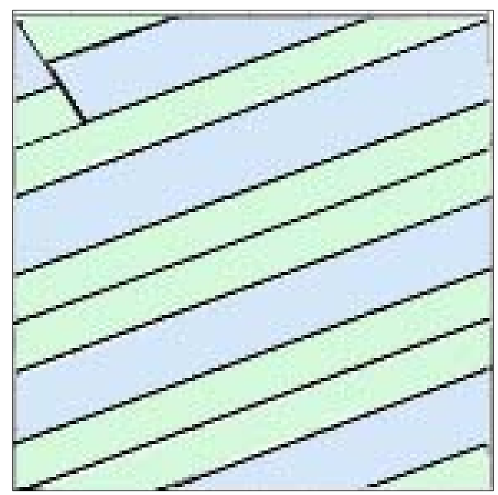


The image of the Markov partition in the plane, above, and folded back into the torus, at right.



Summary

We have found a method that produces two-box Markov partitions for any map on the torus when at least one eigenvalue is positive. When one eigenvalue is negative, both branches of the manifold corresponding to that negative eigenvalue must be used. When both eigenvalues are negative, a three part partition may be possible but probably not a two part partition. This has been demonstrated, but it must be proved.

We would also like to understand what kinds of partitions can be found for maps on the square sphere, in which points on an edge are identified with their reflection about the edge's midpoint. So far it seems difficult to find partitions of less than four parts.