# Network Structure and the Path to Synchronization

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## Abstract

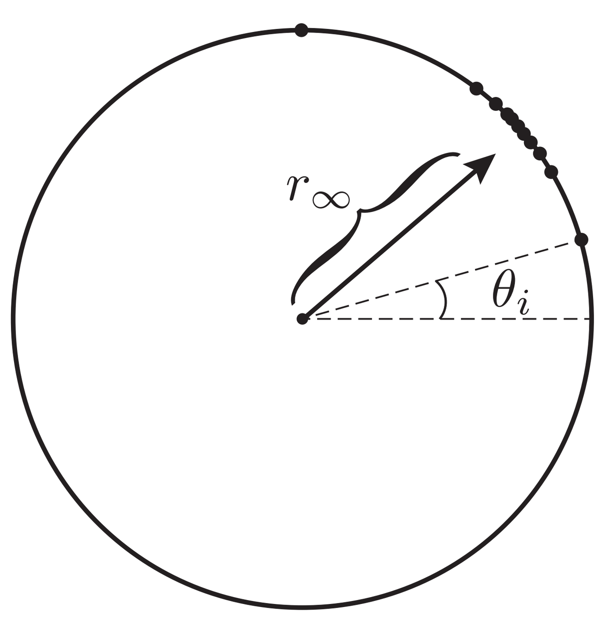
The synchronization of coupled oscillators is a ubiquitous phenomenon occurring in many natural systems. We explore how the network structure that connects the oscillators influences the synchronization process. Specifically, we compare the path to synchronization in homogeneous and heterogeneous networks by evaluating both global and local synchronization measures as a function of coupling strength. We also explore the effects of different natural frequency distributions.

## The Model

To study this problem, we make use of a modified Kuramoto model[1], in which the oscillators are connected via a network:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i)$$

- $\theta_i$ - The phase of the i[th] oscillator
- $\omega_i$ - The natural frequency of the i[th] oscillator. PDF: $g(\omega)$
- $K$ - Coupling Constant
- $N$ - Number of Oscillators
- $a_{ij}$ - Adjacency Matrix




We use the order parameter $r_\infty$ as a measure of oscillator synchronization.

$$r_\infty \equiv \left| \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i} \right|$$

$r_\infty$ ranges from 0 to 1, with $r_\infty \to 1$ as the system becomes perfectly synchronized.

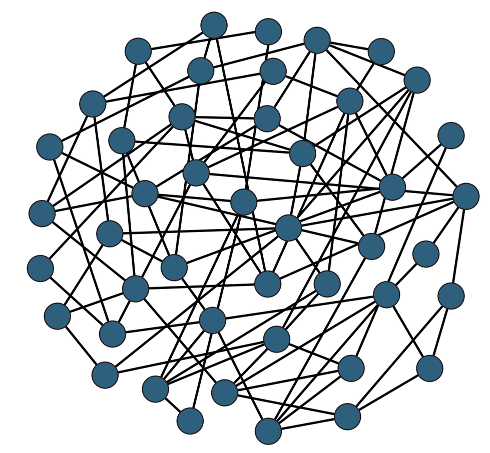
We use the parameter $r_{link}$[2] to characterize the extent of local synchronization in the network.

$$r_{link} \equiv \frac{1}{2N_l} \sum_{i,j} \left| \frac{1}{T} \int_t^{t+T} a_{ij} e^{i(\theta_i - \theta_j)} dt' \right|$$

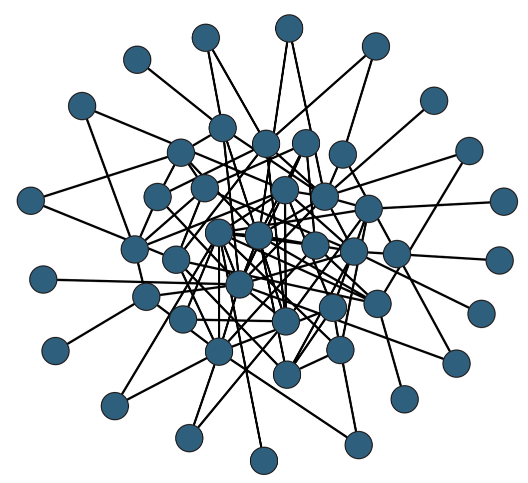
$N_l$ - Number of Links

## The Networks

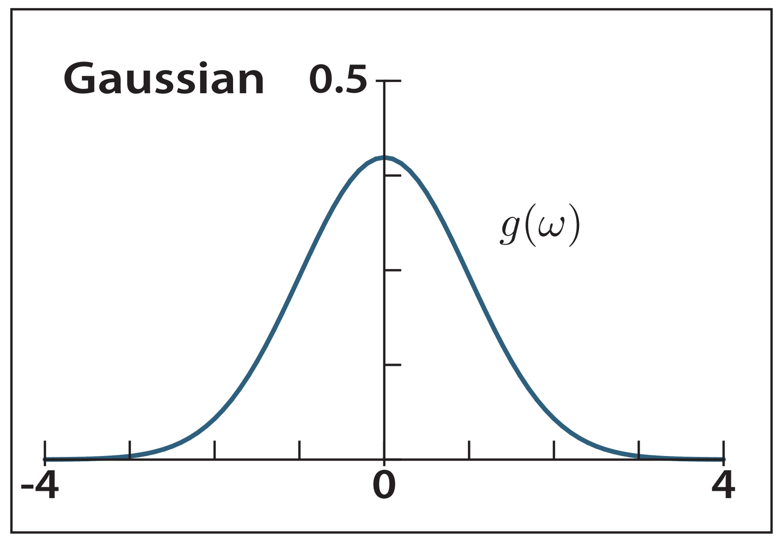
**Erdős–Rényi**[3,4] (ER) networks are homogeneous random graphs in which every possible edge is created with the same constant probability, resulting in a Poisson distribution of degree (number of connections per node).

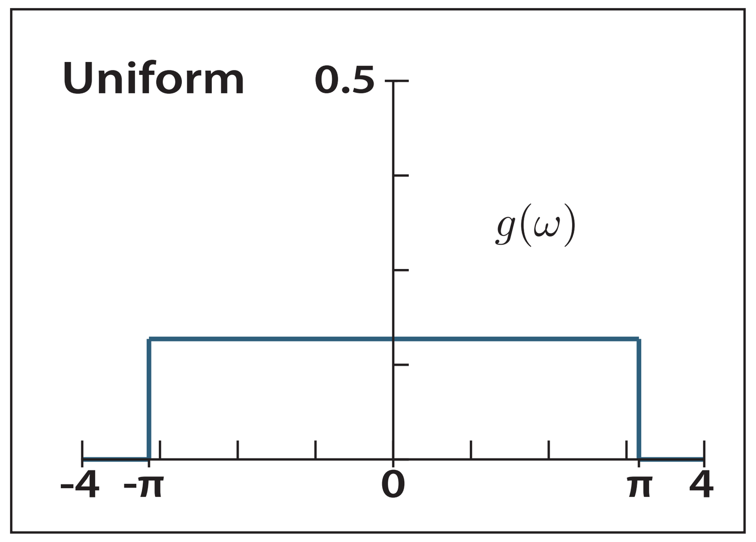


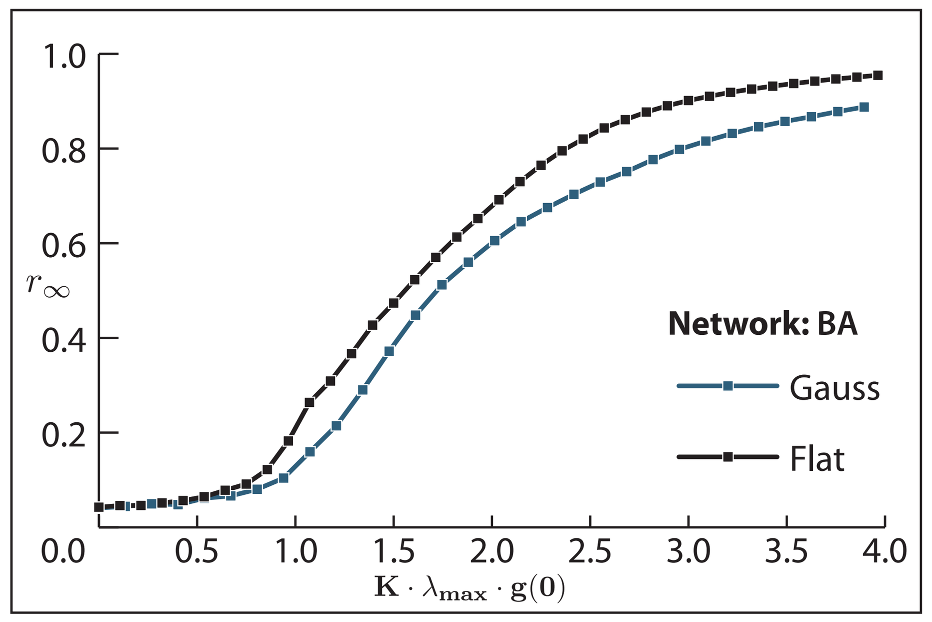
**Barabási-Albert**[4,5] (BA) networks are heterogenous networks created by a growth process in which newly added nodes preferentially connect to existing nodes of high degree. BA networks have a power law degree distribution.



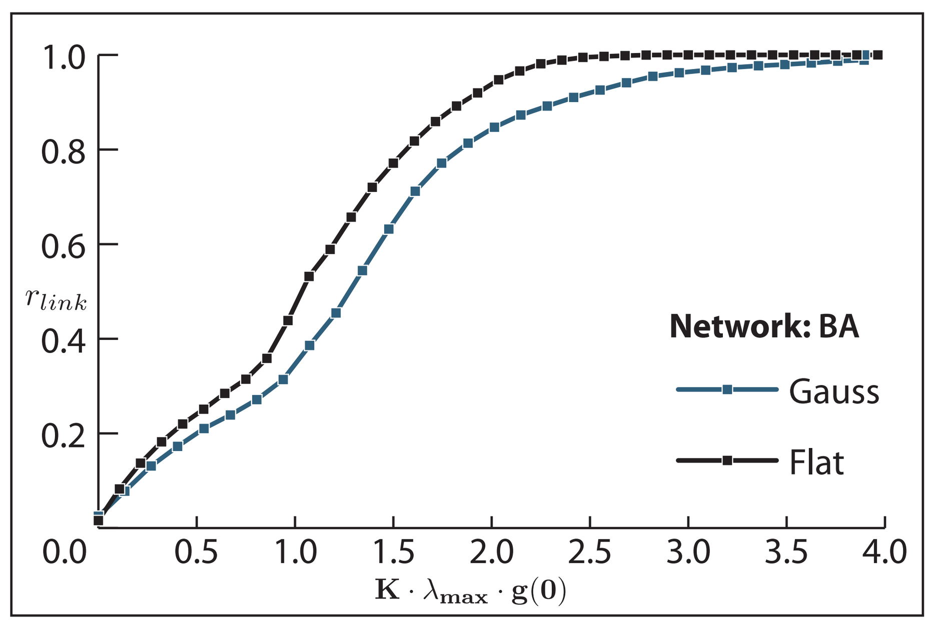
## Varying Frequency Distribution



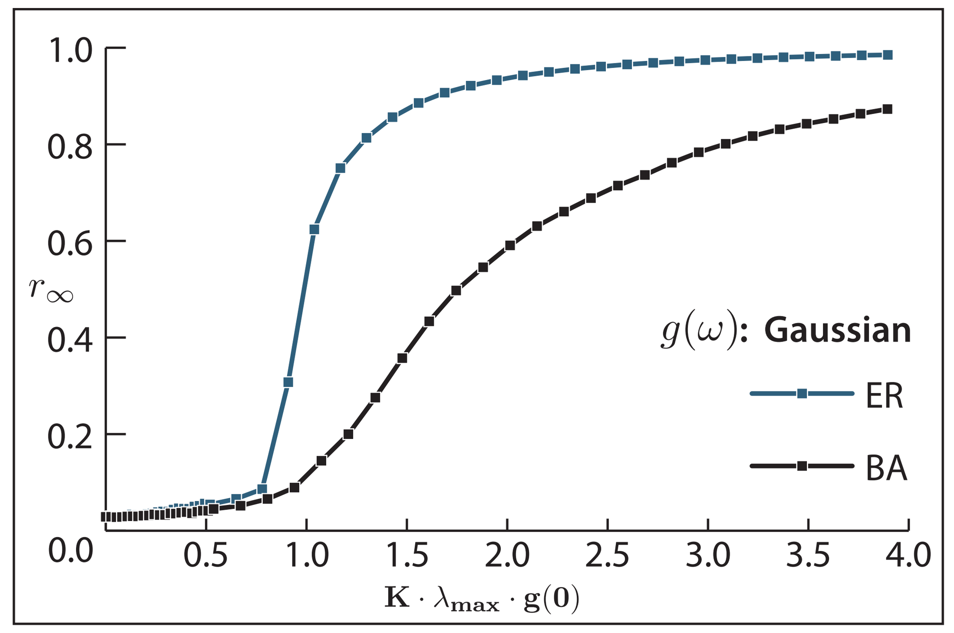

When the natural frequencies of the oscillators are distributed uniformly in $[-\pi, \pi]$, the system exhibits a slightly stronger degree of global synchronization than in the Gaussian distributed case.



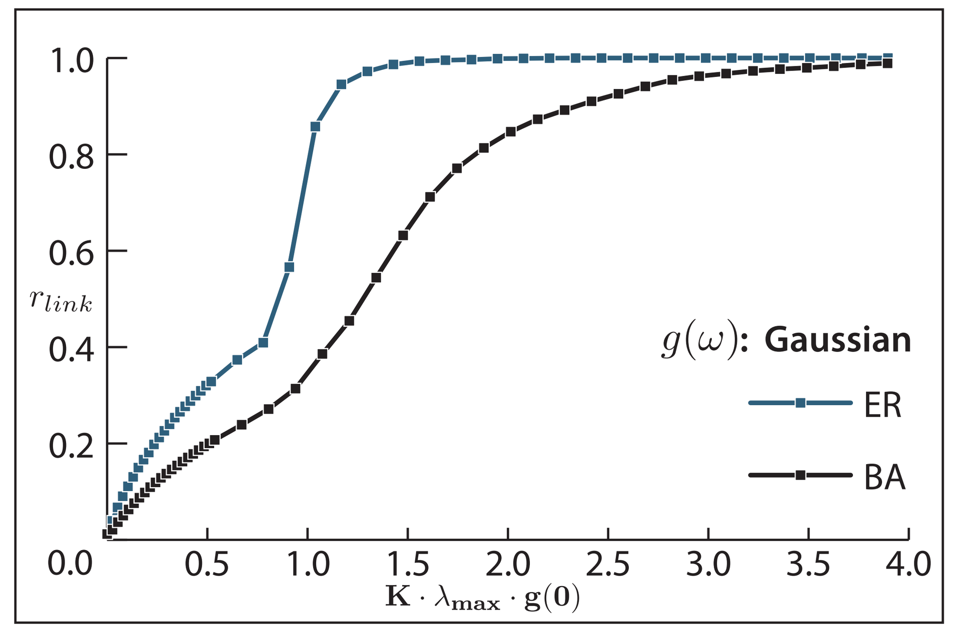

Similarly, we observe a slightly higher tendency for local synchronization with the uniform distribution of natural frequencies.

## Varying Network Structure

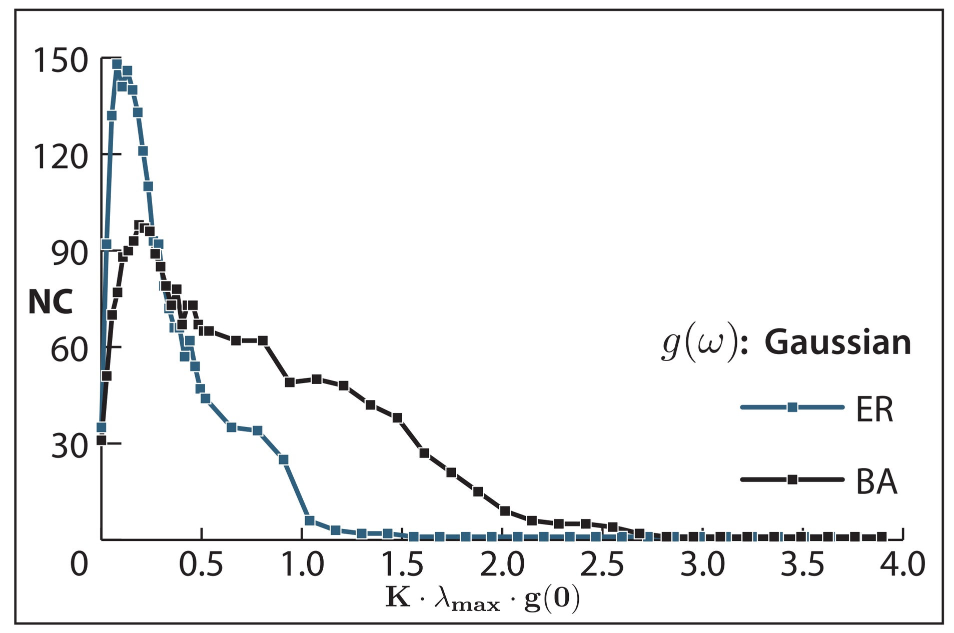
Once the two networks pass the onset point for global synchronization, the homogeneity of the ER network allows it to reach widespread synchronization much more quickly than the BA network.




Both networks exhibit some degree of local synchronization before the phase transition to global synchronization, with the ER network exhibiting a higher degree of local synchronization.




Looking at the number of locally synchronized components, we see that in comparison to the BA network, the ER network forms many small locally synchronized components that quickly join together into a single globally synchronized component.




## Conclusion

As seen above, the structure of the interaction network has a strong effect on the path to synchronization in systems of coupled oscillators. The homogeneity of the ER network causes it to reach a higher degree of both local and global synchronization faster than the heterogeneous BA network. Although frequency distribution does not have a profound impact on the system's dynamics, a uniform natural frequency distribution tends to synchronize both locally and globally earlier than a Gaussian distribution.

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