Explosive Percolation in Directed Networks

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Motivation

- Directed networks can simulate various real world networks, such as gene regulatory networks and the World Wide Web.

- "Explosive percolation" is a new type of percolation transition (described below) being studied on various networks. We study the possible existence of explosive percolation in **directed** networks.

Background

Directed and Undirected Networks

- Directed networks consist of directional links while undirected networks have undirected links.

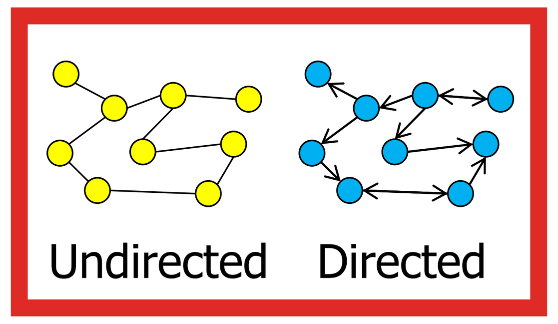


Figure 1. Difference between directed and undirected networks.

What is Network Percolation?

- Percolation is the formation of a macroscopic connected component in a network.

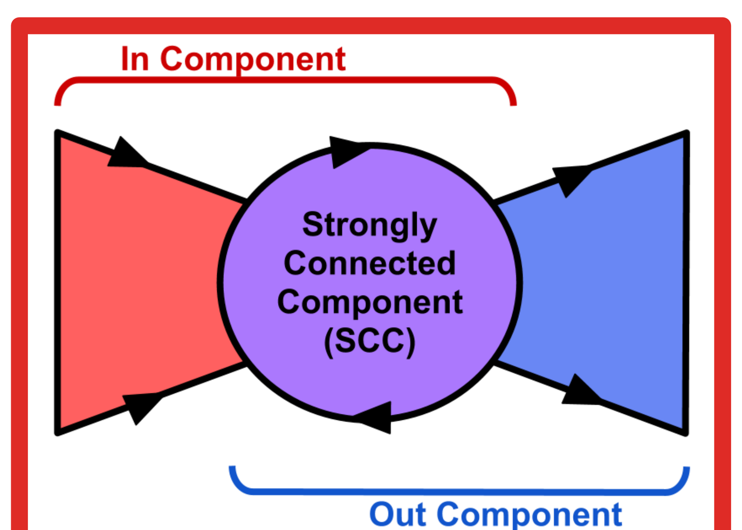
- Randomly adding links between N nodes leads to a continuous percolation transition as N→∞.

What is Explosive Percolation?

- By using a competitive rule for forming links on **undirected** networks, it was recently shown that the transition point can be delayed and the transition appears discontinuous[1].

- Recent analysis[2] shows that it is continuous as N→∞ but appears discontinuous at large but finite N.

Connected Components in Networks



- In undirected networks, any set of connected nodes is a cluster.

- In directed networks, components are described as in Figure 2.

Figure 2 (left). The directed network's components.

Competitive Percolation in Directed Networks

- Randomly choose two pairs of nodes (**a**,**b**) and (**c**,**d**) as candidates for linking.

- Find the total number of nodes that can reach **a** and **c** respectively. These are the *in components* of **a** and **c**.

- Find the total number of nodes that can reach **b** and **d** respectively. These are the *out components* of **b** and **d**.

- Multiply the *in component* and *out component* for each pair.

- Connect the pair (either **a** →**b** or **c** →**d**) with the smallest product, (*in component*)x(*out component*), or randomly if the products are equal.

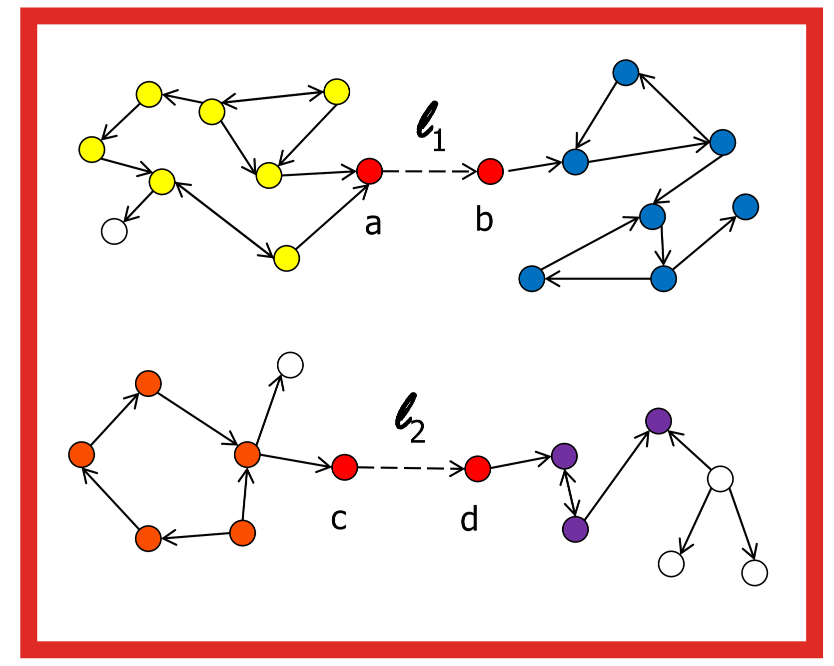


Figure 3. The product for ℓ_1(7x7) is greater than ℓ_2(5x3). Link ℓ_2 is added to the network.

Results

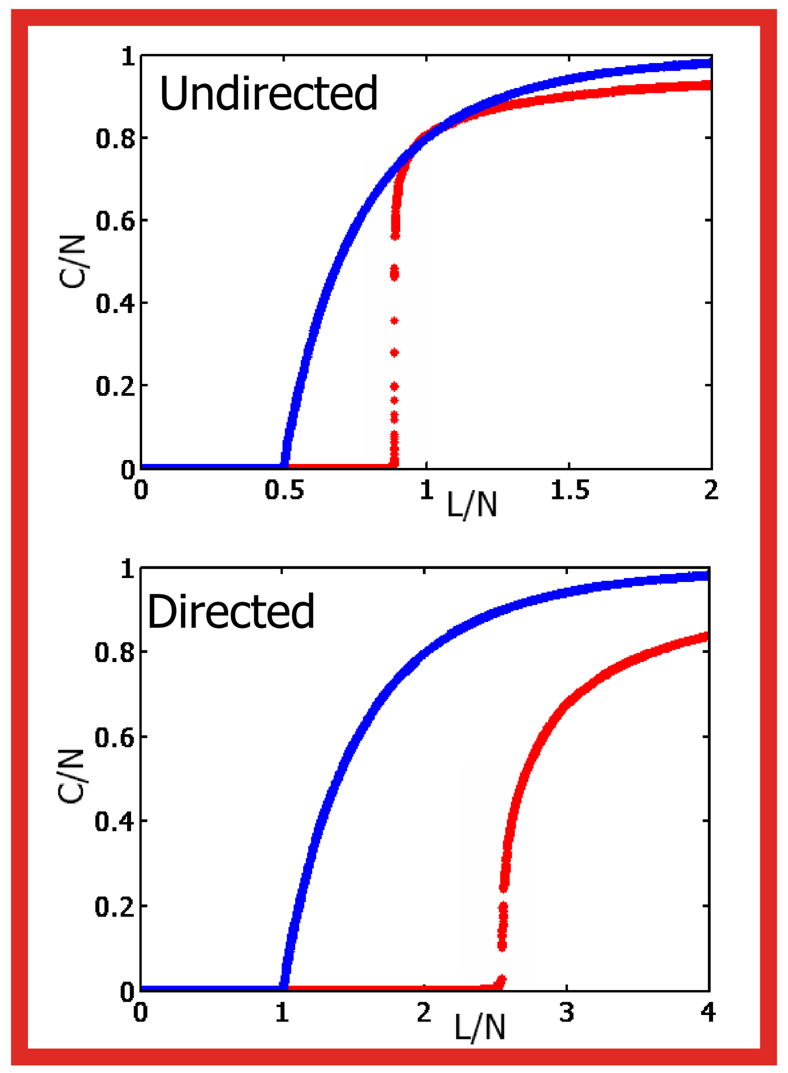


Figure 4. $\frac{C}{N}$ vs. $\frac{L}{N}$ for an undirected network (top) and a directed network (bottom) comparing a random process (blue) with a competitive process (red). Each graph is of one run for one network of size 1,280,000.

- We define p_c as the percolation transition point, and C as the size of the largest cluster.

- For directed networks, C is the size of the out component of the largest SCC, or GOUT. Similar results (not shown) were found for GIN (the in component of the largest SCC).

- Links were added until the total number of links (L) was 2N for undirected networks or 4N for directed network.

- For both directed and undirected networks, we compare the random case and the competitive product rule.

- The competitive rule on directed networks has a less explosive transition than the same rule on undirected networks even though its transition point is delayed even further.

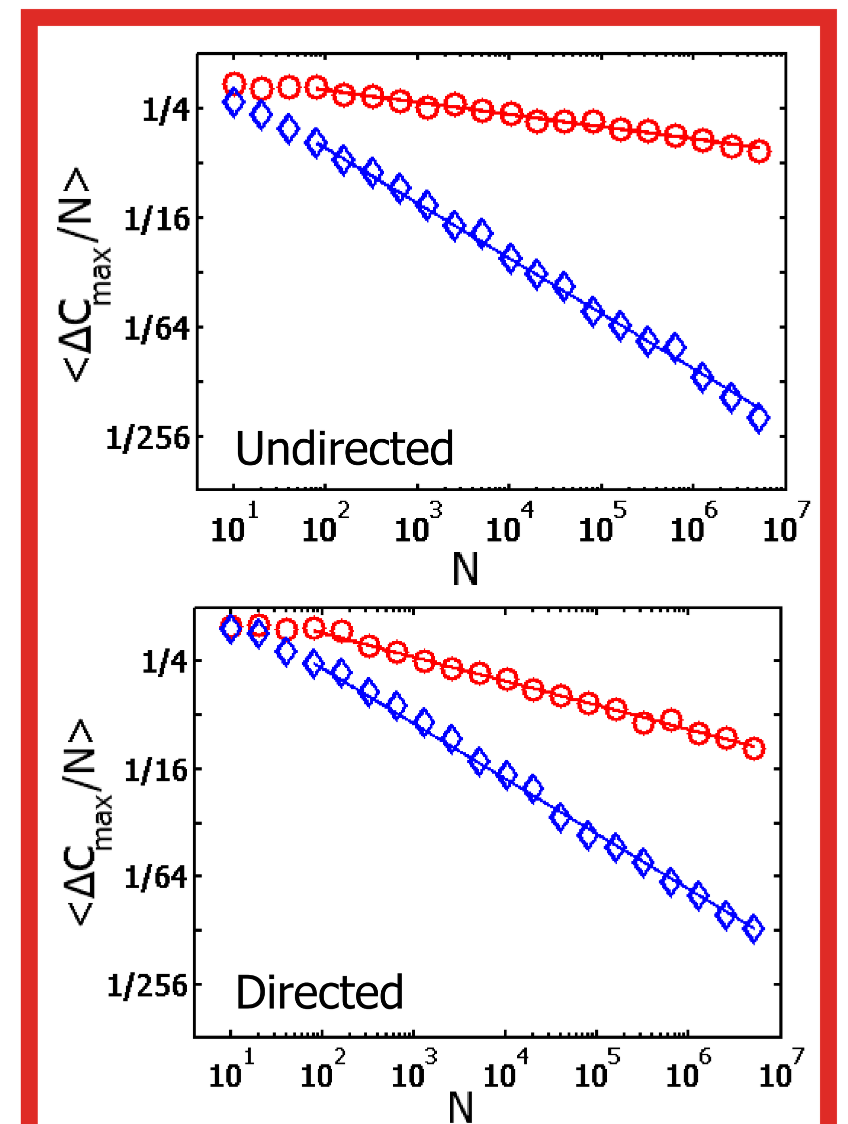


Figure 5. $\left\langle \frac{\Delta C_{max}}{N} \right\rangle$ vs N for undirected networks (top) and directed networks (bottom) on a log-log scale for the random (blue) and the competitive case (red). Each point is the average over 50 networks, and error bars are smaller than the marker size.

Results

- To quantify explosiveness, we also measured the maximum change in C as one edge is added, or ΔC_{max}

- $\frac{\Delta C_{max}}{N} \to 0$ as $N \to \infty$ indicates a continuous transition. (Figure 5)

- Figure 5 shows that the ΔC_{max} jump of the competitive process decays algebraically with N

$$\frac{\Delta C_{max}}{N} \sim N^{-\gamma}, \gamma > 0$$

- Best fit lines determine that the exponent for the competitive undirected case is γ=0.067, while for the random undirected γ=0.303. For the competitive directed case γ=0.135, while for the random directed case γ=0.308.

- For undirected networks the competitive exponent γ is about 5 times less than the random case. For directed networks, the competitive exponent is only half that of the random case.

Conclusion

Using a competitive product rule, a directed network can achieve explosive percolation for its giant in and out components. The maximum jump size decays quicker than the undirected case, suggesting a less explosive percolation transition.

References

[1] D. Achlioptas, R. M. D'Souza, and J. Spencer, Science 323, 1453 (2009).

[2] Grassberger, P., Christensen, C., Bizhani, G., Son, S.-W., & Paczuski, M, . *Phys. Rev. Let.,106,. 225701* (2011).