### ABSTRACT

| Title of dissertation:    | CHAOTIC DYNAMICS                                 |
|---------------------------|--|
|                           | OF A LASER                                       |
|                           | WITH FEEDBACK                                    |
|                           | Ryan Glen McAllister, Doctor of Philosophy, 2003 |
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A diode-pumped solid-state laser is constructed. The complex dynamics resulting from electronic feedback of the laser intensity are explored and characterized. The laser is further perturbed by periodic, quasiperiodic, and chaotic drives.

Distinct stable and chaotic regimes can be elicited from the laser by tuning the bias of the feedback loop. An additional branch of the feedback loop, containing a derivative filter, provides access to new kinds of dynamics, including a more gradual transition to chaos. The whole feedback network together allows the laser dynamics to be selected from among a wide range of chaotic waveforms distinguished by statistical or spectral information. In other words, this laser system can be used as a tunable generator of chaotic functions.

The laser with feedback is perturbed by a signal composed of one or two si-

nusoids. The phase of the chaotic intensity signal is constructed using the complex analytic signal. With one-frequency driving, the laser dynamics show phase entrainment for sufficient driving amplitude. Competition between two distinct driving frequencies to phase synchronize the intensity dynamics of a chaotic laser is observed in the case of two-frequency driving. Competing frequencies alternately show phaselocking and phase-slipping. Competition is quantified by calculating the portion of time the laser phase-locks to each of the driving frequencies and their average.

The electronic feedback signal driving the chaotic dynamics of a laser is recorded. Feedback is removed and the recorded signal is replayed to the laser. Instead of repeating the original dynamics, the laser displays generalized synchronization to its previous waveform. Generalized synchronization error is minimized for slightly mismatched laser parameters. In addition, the generalized synchronization includes phase synchronization between drive and response with synchronized phase slips.

A physical mechanism for generalized synchronization relying on hidden variables of the laser is proposed. A model including this mechanism is investigated and shows good agreement with the experiment.

### CHAOTIC DYNAMICS OF A LASER

### WITH FEEDBACK

by

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Advisory Committee:

Professor Rajarshi Roy, Chair/Advisor Professor Brian Hunt Professor Daniel P. Lathrop Professor Edward Ott Dr. Ira B. Schwartz © Copyright by Ryan Glen McAllister 2003 Perhaps the best thing you can give a person is a pretty good start. I would like to thank Jill, Richard, Jocelyn, and RJ McAllister, and William W. and Betty Sunderlin for that start in terms of family. I owe much gentleness in the early years of my life to my extended family and family friends including Mr. and Mrs. Crawford, Jerry Manning, Judy and Dwayne Singley, Uncle Bill and Aunt Sybill, Aunt Fran and Uncle Chuck, Linda, Leslie and Mema, Janus and Mr. and Mrs. Hart, and Julie Cobern. I owe many thanks also to the family I do not yet know.

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### Chapter 1

### Introduction

Beginning in the earliest days of lasers, and continuing into today a common research goal has been to achieve maximally stable laser output. The scientists and engineers working toward this goal may be shocked and dismayed to discover that today, research toward producing and employing laser instabilities and chaotic dynamics is flourishing.

It turns out that chaotic behavior is not all bad.

In many cases, unstable and chaotic dynamics can be disastrous from a human perspective. Yet it is also becoming recognized that the proper operation of many systems, including the brain [1], require a degree of chaotic-appearing dynamics.

Chaotic dynamics are occuring everywhere around us. On meteorological, ecological, biological, chemical, and physical scales, understanding chaos seems to be imperative for understanding even fundamental phenomena. And perhaps further ahead lies an understanding of some social, economical, and psychological events within the conceptual structure of chaotic dynamics.

Beyond its inherence and necessity in physical systems, chaos may have many additional new applications. Though the term chaos was only introduced in 1975 [2], applications have already arisen or been proposed in areas as diverse as medicine, communications, and material engineering.

The relationship between chaotically behaving lasers and the field of nonlinear dynamics is bidirectional. Not only may we find application of nonlinear dynamical

tools and concepts to understanding laser dynamics, but we may also find applications of laser dynamics to exploring the phenomena of nonlinear dynamics.

### 1.1 Chaos

The linguistic roots of the word "chaos" stretch back beyond the golden age of ancient Greece. Then, chaos may have been portrayed as a formless, seething void, or as a lack of order, or even as the first god, from whom the rest of creation sprung. But now, the technical meaning of chaos has evolved to represent an entirely different concept.

A working definition sufficient for our purposes, found in Ref. [3], is:

**Chaos** is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Chaotic dynamics may, because of the aperiodicity and the sensitive dependence on initial conditions (and, therefore small perturbations), have the outward appearance of randomness. The behavior of an individual system will also most likely prove difficult to predict. Yet, chaotic dynamics are the result of an underlying ordered process.

Good technical references texts for chaos as a field include Refs. [3, 4, 5]. Within the field of chaotic dynamics, a rich landscape of interesting phenomena have been and are being discovered.

Among these, in this dissertation we will encounter the surprising phenomena that chaotic systems, despite the complexity of their behavior (aperiodicity) and unpredictability (sensitivity to initial conditions), can synchronize to one another. By synchronize we mean that, when two or more chaotic systems are coupled together, the behavior of those systems can display causal relationships. Thus synchronization can exist in simple forms, such as how we push the child on the swing with the same period that the child is swinging back and forth. And synchronization also exists in more complicated forms, such as how the many features of the weather must be related to the Earth's turning, the cycles of energy input from the sun, the tides, the ocean currents, and perhaps the activities of the planet's living beings.

Identical synchronization is the simplest relationship that two systems can exhibit. When identically synchronized, the observable behavior of both systems is the same.

Phase synchronization can be more difficult to observe than identical synchronization as the amplitudes of the behavior of two systems may not have an evident correlation, but a sensibly defined phase does [6, 7, 8]. Phenomena such as tides, the seasons, and sleep are examples of phase synchronization.

Generalized synchronization can be less apparent than either identical or phase synchronization, as it is characterized only by the existence of some functional relationship between the systems in question [9, 10, 11, 12]. This form of synchronization may also be quite common.

An overview of many concepts, examples, and proposed applications of synchronization phenomena can be found in Ref. [13]. A review specifically of the concept of phase synchronization can be found in Refs. [14, 15]. Generalized synchronization has been introduced more recently, and is an exciting area of current research.

#### 1.2 Lasers

Laser design and implementation is an exciting and still-evolving field of research and engineering. In the experiments performed in this dissertation, the laser itself is a research tool as well as the research subject, therefore the technical aspects of the laser design are restricted to those presented in Chapter 2. Good general laser design texts for further reading include Refs. [16, 17].

Many of the optical phenomena relevant to the laser system used in this dissertation are described in modest detail in Chapter 2. Suggested further reading in optics might include Ref. [18], or the more detailed Ref. [19].

Since the first published report of lasing action in 1960 [20], stable lasers have

been used for a variety of applications in areas such as communication, medicine, and materials processing. Stably operating lasers have also furnished a remarkable device for scientific research and science fiction plot lines.

Once we move beyond the view that lasers are only useful when operating stably, laser dynamics and laser chaos becomes a subject of research. Good texts on laser theory, principles, and phenomena include Ref. [21] and the longer Ref. [22]. In 1975 [21, 23], an analogy was drawn between a single-mode laser model and the well-known Lorenz model of convection-driven turbulence.

This analogy gives us a three-part connection: Laser systems can be like hydrodynamic systems can (perhaps) be like other nonlinear systems. Even in the case where systems are not equivalent through some transformation of variables, similar behavior may be observed in each of them. Examples of this similarity in seemingly disparate systems includes the Feigenbaum number involved in the fractal appearance of period doubling bifurcations [24] and the commonality of bifurcations and synchronization of chaos, like that discussed here.

The nonlinear dynamics of this laser system under different conditions of feedback and drive is the subject of this dissertation. It is hoped, though, that these phenomena are sufficiently general to be applicable to a wide variety of natural and engineered systems.

## 1.3 Prior Work in Laser Chaos and Synchronization

It is known that an autonomous, continuous-time system must have three degrees of freedom to exhibit chaos. The time evolution of a generic single-mode homogeneously broadened laser is described by three dynamical variables: the complex electric field, the population inversion, and the complex polarization [21], a sufficient number for it to display chaos.

In a class-B laser, of which the neodymium laser used in this dissertation is

an example, the decay rate of the polarization  $\hat{D}$  is large enough compared to those of the other two variables that the polarization is essentially determined by the instantaneous values of the electric field and inversion (see Chapter 3). Without the addition of other elements, such lasers are not chaotic and are instead often employed for research and industrial applications in either stable (continuous wave) or regularly pulsed modes of operation [16].

For class-B lasers, and other lasers not already chaotic in behavior, several means have been used to generate chaos. External (nonautonomous) driving of the cavity losses [25, 26] and of the cavity length [27] were suggested and explored in the 1980s. Injection of light from another laser can also destabilize laser dynamics [28].

In 1986, a way to destabilize the laser so that it could produce chaotic dynamics without any external perturbation was introduced using an electronic feedback loop and an intracavity modulator [29]. It is this method we will use to produce the chaotic dynamics studied here. Chapter 2 describes the setup in more detail.

Chaotic lasers are convenient systems in which to study the synchronization of nonlinear dynamical systems. Identical synchronization of laser chaos was first reported in 1994. It was reported independently in a pair of unidirectionally coupled CO<sub>2</sub> lasers [30] and in a pair of mutually coupled Nd:YAG lasers [31]. In the following years, experiments such as those in Refs. [32, 33, 34, 35, 36, 37, 38, 39] investigated identical synchronization of one-way coupled lasers and Refs. [31, 40, 41, 42, 43, 44] investigated the case of mutually coupled systems.

Phase and generalized synchronization have become of interest more recently, with Refs. [45, 46, 47, 48] investigating the phase synchronization of laser chaos and Refs. [9, 49, 50] investigating generalized synchronization.

#### **1.4** Organization of this Dissertation

The laser system itself is described in detail in Chapter 2, beginning with the physical properties of the laser itself, proceeding through the design of the feedback loop,

and concluding with some examples of the simplest laser phenomena.

Several models for the laser are described in Chapter 3. Several of the approximations typically used to construct workable equations for simulating laser dynamics are discussed. Two laser models that will be used in the following chapters are presented, a single-mode model and a multi-mode model.

Some of the dynamics seen in the laser with the feedback loop are described in Chapter 4. The utility of the laser as a tunable source of chaotic functions is examined. Statistical and spectral characterization of the laser intensity signals allows selection of a waveform fitting some given criteria. Additions to the feedback loop that allow access to an even wider range of dynamics are presented. The singlemode laser model is used to simulate some of the laser dynamics and is shown to be insufficient for modeling the chaotic laser dynamics.

In Chapter 5, phase synchronization of the laser is investigated. The chaotically operating laser with feedback is subjected to external sinusoidal driving. The Hilbert phase method is used to discriminate the phase of chaotic signals. In the case of driving with a single frequency, the dynamics show alternating phase synchronization and phase jumps. In the case of driving with two frequencies, competition between each frequency and the average for synchronization of the laser signal is shown. While spectrograms and examination of the attractor are inadequate for observing the competition for phase synchronization of nearby frequencies over short time intervals, the staircase plot introduced in Ref. [51] allows easier detection.

In Chapter 6, the feedback signal generating the laser chaos is recorded. Live feedback is removed and the recorded signal is replayed to the laser. Instead of showing identical synchronization to its original dynamics as might be expected, the laser shows generalized synchronization. Synchronization of both the phase dynamics and phase-slipping is observed. An explanation for generalized synchronization is suggested involving the multimode character of the laser. The multimode model is shown to agree with the experiment. The surprising result that generalized synchronization is best for slightly mismatched feedback loop bias is observed in both the experiment and the model. Chapter 7 contains the concluding remarks.

### Chapter 2

### **Experimental Design**

#### 2.1 The Laser System

The following experiments use a diode-pumped solid-state laser. The laser is referred to as "solid-state" because the lasing medium is a crystal. The laser cavity is formed by two mirrors; one is coated onto a surface of the crystal and the other is an independent spherical mirror. The laser is called "diode-pumped" because energy is optically pumped into crystal by a diode laser. The pump operates at a wavelength that the crystal can absorb, and photons are re-emitted by the crystal at its lasing wavelength. Both of these lasers emit light in the near infrared portion of the electromagnetic spectrum.

The laser crystal, neodymium-doped yttrium aluminum garnet (Nd:YAG), consists of a substrate,  $Y_3Al_5O_{12}$ , and a dopant, Nd. The crystal is doped at approximately 1%, meaning that at approximately 1 out of every 100 Yittrium lattice sites, the Y<sup>+3</sup> ion has been replaced by an Nd<sup>+3</sup> ion. The lasing transition occurs within the neodymium ions.

The Nd:YAG laser is a four-level laser. The lasing transition occurs between the two middle levels (2 and 3), when an atom in state 3 de-excites to state 2 by emitting a photon. The ratio between the number of atoms in state 3 and in state 2 is called the population inversion. The upper lasing level has a fluoresce lifetime of ~230  $\mu$ s at 1% doping. The lower lasing level decays to the ground state via a nonradiative transition with a lifetime of  $\sim 30 \ \mu$ s. The transition from ground state (1) to upper level (4) in Nd:YAG corresponds to an absorption band centered near 810 nm. The lasing transition from 3 to 2 corresponds to an emission band around 1064.1 nm.

The crystal is a cylinder with both faces polished flat. One face is coated with a dielectric mirror highly reflective at the lasing wavelength (1064 nm) and highly transmissive at 810 nm. The other crystal face is antireflection coated for 1064 nm to minimize reflections of the light inside the cavity by objects other than the laser mirrors. The index of refraction of Nd:YAG at 1064 nm is 1.82. This and additional information about Neodymium-doped YAG lasers can be found in Ref. citeKoechner.

The output coupler of the laser is a spherical mirror with a radius of curvature of 10 cm. It has a dielectric coating with a transmissivity T of 2% at 1064 nm. During the course of these experiments, the distance between the output coupler and the laser crystal varies between 7 cm and 9 cm. Thus the cavity always satisfies the condition for a stable laser resonator with one flat and one spherical mirror. Namely, the distance between the two mirrors is less than the radius of curvature of the spherical mirror. The optical length of the cavity is slightly greater than the distance between the reflective surfaces of the crystal and output coupler because of the high index of refraction of Nd:YAG.

Population inversion in the solid-state laser is achieved by diode laser pumping at 810 nm through the coated face of the crystal. The beam from the diode (SDL model 2350H1) is initially highly divergent and astigmatic. Its divergence is reduced with a Newport F-L20 telescope objective (0.3 numerical aperture). The pump beam then passes through two plano-convex cylindrical lenses, with focal lengths 19 mm and 80 mm placed 99 mm apart to reduce the astigmatism. The pump beam is now approximately a rectangle measuring 90 mm by 80 mm. Finally, the beam is focused down to a small waist inside the crystal by a spherical lens with a focal length of 75 mm. All three lenses are antireflection coated for the pumping wavelength of 810 nm. The pump optics are shown in Fig. 2.1.



Figure 2.1: Optics shaping the pump beam for the Nd:YAG laser. The horizontal line represents the pump beam center line. The plano-convex cylindrical lenses have focal lengths  $C_1 = 19$  mm and  $C_2 = 80$  mm. They are placed with the curved faces toward the more collimated beam (away from each other). The spherical lens has a focal length of 75 mm and focuses the beam to a small spot within the laser crystal.

The pump laser sits in an aluminum mount with a thermistor, thermoelectric coolers (TECs), and a water-cooled heat sink. A Spectra Diode Labs SDL-800 controller is used to maintain the diode laser temperature at 23.5  $\pm 0.3$  °C. The SDL-800 controls the output power of the laser to a precision of 1 mW.

The cavity is aligned such that the Nd:YAG laser beam operates in the Gaussian transverse electromagnetic mode  $\text{TEM}_{00}$ . The lowest pump power for which the laser will lase is called the laser threshold. The laser threshold is minimized by refining the cavity alignment, specifically by tilting the output coupler and moving the focal point of the spherical lens.

An Isomet 1205C-2 acousto-optic modulator (AOM) is inserted into the cavity between the crystal and the output coupler. This requires realignment of the cavity, and the laser threshold is increased due to the absorption and scattering losses within the AOM. The output coupler tilt, the focal point of the spherical lens, and the position and tilt of the AOM are all adjusted to minimize the laser threshold with the AOM in the cavity.

The AOM is a lead molybdate (PbMoO<sub>4</sub>, also known as wulfenite) crystal with a piezoelectric transducer (PZT) affixed to one face. An Isomet model 232A AOM driver causes the PZT to vibrate at a radio-frequency (RF) carrier frequency  $f_{RF} = 80$  MHz, and the amplitude of the vibrations is determined by the voltage applied to the video input of the driver.

The AOM allows an electrical signal to deflect a portion of the lasing beam and therefore increase the cavity loss. The acoustic waves generated within the crystal by the PZT create a phase grating through which the light in the cavity passes. The speed of sound through the AOM crystal is  $v_{AOM} = 3.63$  km/s. Thus the spacing of the diffraction grating is  $v_{AOM}/f_{RF} = 45 \ \mu$ m. The light in the undeflected 0<sup>th</sup> order beam remains within the stable lasing cavity.

With the AOM in place, the optical length L of the cavity is increased. Lead molybdate is birefringent, meaning that the effective index of refraction varies depending on the polarization of the incident light. At 1064 nm, Isomet reports the two indices of refraction are  $n_o = 2.22$  and  $n_e = 2.36$  for light shown down the a-axis. The crystal in the AOM is a-cut, meaning the laser beam shines down the a-axis. The length of travel of light within the AOM is approximately 1 cm, so insertion of the AOM increases the optical length of the cavity by approximately 1.3 cm if the output coupler is not moved.

The laser alignment is very sensitive to small vibrations. The laser is constructed on an optical table with air legs that dampen vibrations coming from the ground. The laser is so sensitive that airborne currents and vibrations also affect its alignment. A cardboard box with small holes cut for entry of the pump beam and exit of the lasing beam is placed around the laser cavity to protect it from these vibrations.

#### 2.2 Detection of the Laser Signal

A laser field has several degrees of freedom. Of these, the intensity of the laser is the easiest to measure. Measurements of the laser intensity are obtained with a photodiode connected to a Liconix 40D optical detector. Light absorbed by the photodiode generates a voltage inversely proportional to the intensity of the absorbed light.

The photodiode circuit has a capacitance that was measured to be around

 $C_{PD} = 300$  pf and the output voltage is measured across an  $R = 500 \ \Omega$  resistor. Thus the response of the photodiode is like a low-pass filter with bandwidth  $(2\pi RC)^{-1} = 1$ MHz. The pump beam is blocked from the photodiode by placing a 1064 nm bandpass filter between the laser and the photodiode.

While the cavity and crystal are aligned such that the YAG laser operates in the  $\text{TEM}_{00}$  mode, the laser can still be composed of multiple longitudinal or polarization modes. In the range of the chaotic oscillations around which most of the following discussions center, the cavity typically lases in two or three modes, with more modes occurring for higher pump power.

Cavity laser modes are standing waves of the electric field. The electric field of any cavity mode of the laser is required to have a node at a mirror face. This requirement is equivalent to the boundary condition

$$L = \frac{N\lambda}{2} = \frac{Nc}{2f} \tag{2.1}$$

where L is the optical length of the cavity, *lambda* is the lasing wavelength, N is the integer number of half-wavelengths fit between the cavity mirrors, f is the lasing frequency, and c is the speed of light.

Distinct longitudinal modes differ by an integer number of half-wavelengths over the length of the cavity, as depicted in Fig. 2.2. The spatial overlap between the electric fields of each mode and the population inversion within the laser crystal are therefore different. A model including the effect of two longitudinal modes is presented in Chapter 3 on page 29.

The frequency distance between adjacent longitudinal modes is called the free spectral range (FSR). The FSR of the laser cavity is c/(2L). In this cavity the FSR is approximately 1.3 GHz, corresponding to a difference in wavelength of approximately 5 pm at 1064 nm.

As the laser does not contain any strongly polarizing elements within the cavity, the laser field may consist of a single polarization or of two orthogonal polarizations. Due to the birefringence of the lead molybdate crystal in the AOM, orthogonal polarization modes with the same number of wavelengths within the



Figure 2.2: A cartoon rendering of the electric field oscillations of two longitudinal modes in the laser cavity. The green mode fits several more half-wavelengths than the purple mode between the mirrors. The frequency difference between the purple and green modes is roughly equal to the free spectral range of the cavity times the difference between the number of half-wavelengths each mode fits in the cavity.

cavity will also have different optical frequencies.

The mode structure of the Nd:YAG laser can be observed with a Burleigh scanning confocal Fabry-Perot interferometer. The interferometer has a free spectral range of 2 GHz, meaning that optical frequencies differing by more than 2 GHz will appear in the scan with their frequency differences modulo 2 GHz. A typical Fabry-Perot display is presented in Fig. 2.3.

The 11,450 sample distance between the duplicated spikes A and D corresponds to the 2 GHz free spectral range of the interferometer. Spike A is centered at 507 samples, spike B is centered at 810 samples, and spike C is centered at 7,793 samples. Thus the frequency difference between the modes represented by Spikes A and B modulo 2 GHz is 52.9 MHz. The frequency difference between the modes represented by Spikes A and C modulo 2 GHz is 1.27 GHz. Because the FSR of the Fabry-Perot interferometer is similar to that of the cavity, it is difficult to determine much more than the number of modes within the cavity.

The photodiode signal is recorded with a digital oscilloscope (CompuScope CS1450) capable of acquiring 50 Msamples/s at 14 bits of precision. Measurements



Figure 2.3: A typical signal from the scanning confocal Fabry-Perot interferometer. The spike A at the left-hand side (centered at 507 samples) and the spike D at the right-hand side (centered at 11,967 samples) represent the same laser modes. The distance between the pair of spikes (11,460 samples) represents 2 GHz. There are three distinct modes in the cavity (A, B, and C).

are taken at either 2 or 5 Msamples/s to provide for sufficient resolution of the time series.

#### 2.3 Design of the Feedback Loop

The laser on its own, as just described, operates at a stable output intensity for a given pump power and a DC voltage applied to the AOM driver. When the laser is described as chaotic, this steady state has become unstable and the laser intensity fluctuates in time. Though much work in the early days of lasers was devoted to making their behavior as stable as possible, it is our intention to make the laser behave chaotically.

A simple way to destabilize a laser without an external drive is through the use of an electronic feedback loop. This configuration was proposed for the first time in 1986 for a  $CO_2$  laser with an intracavity electro-optic modulator (EOM) [29]. Dynamics of a  $CO_2$  laser with electronic feedback have been further investigated in Ref. [52]. Another method that may be used to induce chaos in a laser is modulation by an independent driving signal, often a periodic signal. Theoretical investigations of this method include Refs. [61, 53]. The dynamics of the laser with feedback, external driving, and a combination of both will be explored here.

In the laser system we study, either the pump parameter or cavity losses can be modulated at rates of several MHz. However, the decay rate of the electric field in the cavity is much faster than that of the population inversion, see Chapter 3. Loss modulation is therefore more effective at influencing the intensity dynamics of the laser than pump modulation.

Compared with EOMs, acousto-optic modulators (AOMs) operating in the visible or near-infrared offer the considerable advantages of 1) a lower driving voltage than EOMs, and 2) not requiring the use of intracavity polarization elements (Brewster windows, gratings, etc.). For this reason, the voltage of the feedback network will be used to modulate the cavity loss via an AOM within the laser cavity.

The sound field in the AOM crystal is thick enough that there are essentially
only two diffraction orders, the zeroth-order beam (undeflected) and the first-order beam. The cavity loss induced by the sound field within the modulator is that amount diverted into the first-order beam, which is

$$\sin^2\left(\frac{\pi V}{V_{MOD}}\right) \tag{2.2}$$

where V is the voltage applied to the modulation input of the AOM driver and  $V_{mod}$  is the saturation voltage of the driver. With a DC signal applied to the AOM driver, the intensity output of the laser is stable and depends upon cavity alignment and pump power.

Feedback can be added to the system by applying the voltage from the photodiode to the AOM driver through an amplifier/attenuator. Feedback with sufficient amplitude and bias destabilizes the steady state laser intensity and gives rise to a variety of dynamical behaviors, including chaotic spiking and multiple-spike bursting.

Connecting the output of the photodiode directly to the input of the AOM driver poses difficulties when one also wants to accurately measure the output voltage of the photodiode. The AOM driver transmits back along the input line the 80 MHz RF carrier that is driving the AOM. At measurement speeds of 2-5 Msamples/s, this appears as white noise of amplitude proportional to the AOM signal.

Insertion of a Stanford Research Systems SR560 Low-Noise Preamplifier reduces the back propagation of the AOM RF carrier. The preamplifier has two inputs and so also allows for addition of another input into the feedback loop. The SR560 has adjustable gain and a filter feature with adjustable bandwidth and 6 or 12 dB/octave rolloff. Most of these experiments are performed at unity gain, bandwidth  $\beta = 100$  kHz, and 12 dB/octave rolloff.

The second input of the SR560 is used to add a bias control, as well as any external modulation, to the feedback loop. The effects of the bias control will be explored in Chapter 4. Bias may applied to the feedback loop by a potentiometer connected to a voltage supply.

Bias and an external modulation signal may also be applied to the laser feed-



Figure 2.4: An example of laser intensity signal (A), the corresponding signal after the pre-amplifier (B), and the AOM driver signal (C) at the same time. The photodiode signal has been inverted to correspond to the laser intensity. All signals have been shifted in voltage to clarify the view.

back loop from a National Instruments NI5411 Arbitrary Function Generator card. This is a computer-controlled card capable of generating voltage signals composed of up to two million samples played at a rate of up to 40 Msamples/s with 12 bit precision.

It was found that noise coming back from the input stages of the preamplifier was interfering with measurement of small photodiode signals. For some experiments an inverting amplifier is inserted between the measurement point and the preamplifier to further reduce the noise detected up by the digital oscilloscope. The noise is thus reduced to the order of 1 mV peak-to-peak.

The laser and feedback loop as just described is shown in Fig. 2.5. Additions and changes in configuration for specific experiments will be discussed in the relevant chapters.

There is a delay between the time when light from the laser falls on the photodiode and when the AOM responds. This feedback loop delay comes in two parts, propagation of the sound wave inside the AOM and transmission time of the electronic signal. Transmission of the electronic signal through the coaxial cables introduces a delay on the order of tens of nanoseconds. The speed of sound within the AOM crystal is 3630 m/s and the distance between the transducer and the laser beam is approximately 5 mm. This corresponds to a time delay of ~1.37  $\mu$ s. Observations of the time delay between a square pulse sent to the AOM driver and the corresponding change in the laser intensity verifies this time delay to be on the order of 1.3  $\mu$ s.

The effect of the time delay on the dynamics is uncertain. However, we do not observe a peak in the power spectrum corresponding to the delay, which indicates that the effect is probably small. This is further confirmed by the fact that the characteristic frequency associated with the delay time is an order of magnitude higher than the cutoff frequency of the feedback loop (f = 100 kHz).

Introduction of additional delays into the feedback loop by inserting an Allen Avionics pulse and video delay line (part No. AV1286) reveals that delays on the order of 10  $\mu$ s do affect the dynamics by introducing new torus-breaking bifurcations.



Figure 2.5: The Nd:YAG laser with intracavity Acousto Optic Modulator (AOM) and feedback loop. The pump beam is absorbed by the optical bandpass filter. The photodiode signal is recorded by the Compuscope card, an ACS1450 Digital Oscilloscope residing inside the computer. The signal can also be displayed on an analog oscilloscope, a Goldstar model OS-9020A, and a digital oscilloscope, a LeCroy model 9310L. Example signals measured at A, B, and C are presented in Fig. 2.4 to illustrate the transformation of the feedback signal through the feedback network.



Figure 2.6: The output power of the Nd:YAG laser as a function of the pump power. The intracavity AOM has already been placed within the cavity, but is turned off. In this case, the laser threshold is approximately 18.5 mW.

#### 2.4 Characteristics of the Experimental System

The laser threshold of this system has been observed to be as low as 11 mW without the additional losses introduced by inserting the AOM inside the cavity. Above threshold, the variation of the output power as a function of pump power is shown in Fig. 2.6. The feedback loop in this case is disconnected and the AOM is off. Still, the static losses introduced by the AOM have increased the laser threshold by over 50%.

Left entirely undisturbed, the laser output intensity would be essentially stable. Vibrations within the room, including sufficiently loud sounds or tapping on the optical table, alter cavity length and alignment, perturbing the laser from the steady state. The laser returns to steady state via damped oscillations know as relaxation oscillations. An example of laser relaxation oscillations is shown in Fig. 2.7.

As can be seen in Fig. 2.7, the smaller relaxation oscillations seem to have a consistent frequency. This relaxation oscillation frequency  $(f_{rel})$  of the laser intensity



Figure 2.7: Relaxation oscillations produced in the laser by tapping on the optical table. In this case, the laser is being pumped at 33 mW.

varies with the pump power, shown in Fig. 2.8 (a). Via the expression derived as Eq. 3.11 on page 28, it is possible to estimate the cavity loss per round trip.

In the next chapter, we will consider some rate equations we might use to model the dynamics of this laser. We will consider both a single-mode model and a two-mode model.



Figure 2.8: a) The variation of the relaxation oscillation frequency with pump power. b) Plotting relaxation oscillation frequency squared vs. the pump parameter D gives a straight line. The pump parameter is defined as the pump power divided by the laser threshold, minus 1.

#### Chapter 3

# Model

### 3.1 Laser Models

I think that laser models are surprisingly accurate given the number of assumptions used to produce them. In this section, we will look at two sets of macroscopic laser equations useful for modeling solid-state lasers. We begging by looking at the Maxwell-Block single-mode laser equations. We derive the lasing frequency and the equations for an Nd:YAG laser from the more general "Class-C" laser equations.

We renormalized the laser equations for analysis and numerical simulation. We derive the steady-state laser solution and explore the phenomenon of relaxation oscillations.

Finally, we present the two-mode Tang-Statz-deMars laser model. It will be shown in Chapter 6 that this model more accurately reproduces the chaotic laser dynamics described in Chapter 4 and the observed phenomena of phase and generalized synchronization in Chapters 5 and 6.

## 3.2 Single-Mode Laser Model

The time evolution of a generic, homogeneously broadened single-mode laser can be described by three macroscopic dynamical variables: the complex electric field, the population inversion, and the complex polarization [21]. Among the many forms of

the class C laser equations, these dynamical variables have been modeled by

$$\frac{\delta E(x, y, t)}{\delta t} = -(i\omega_c + k_E)E + kP + i\alpha k\Delta_{\perp}^2 E,$$
  

$$\frac{\delta P(x, y, t)}{\delta t} = -(i\omega_A + \gamma_{\perp})P + \gamma_{\perp} ED,$$
  

$$\frac{\delta D(x, y, t)}{\delta t} = -\gamma_{\parallel} \left[ (D - D_0) + \frac{1}{2} (E^*P + EP^*) \right].$$
(3.1)

In this form of the class C laser equations, E is the complex-valued electric field, P is the atomic polarization of the crystal, and D is the population inversion of the gain medium.  $k_E$  is the decay rate of the electric field within the cavity, which accounts for reflectivity of the cavity mirrors and any scattering or absorption (without reemission) within the cavity. The equations thus expressed can be found in [54].  $\gamma_{\parallel}$ and  $\gamma_{\perp}$  are the decay rates of the population inversion and the atomic polarization respectively.  $D_0$  is the pump parameter.  $\omega_C$  and  $\omega_A$  are the resonance frequencies of the cavity and lasing atoms in angular frequencies.  $\alpha$  is the coefficient of diffraction.

$$\alpha = \frac{1}{2\pi FT},\tag{3.2}$$

where T is the transmissivity of the cavity mirrors and  $F = b^2/\lambda L$  is the Fresnel number of the cavity.  $\lambda$  is the wavelength of the laser light, L is the optical length of the cavity, and b is the diameter of the active region in the crystal.

The laser variables are presented in Eq. (3.1) above as scalar functions of time and position within the gain medium. Already several assumptions and approximations have been therefore made, including representation of the electric field as a scalar.

However, spatial averaging of E, P, and D can lead to significantly easier computation while retaining surprisingly accurate analytical and numerical predictions of laser behavior. When eliminating the spatial variation of the laser variables, the  $\Delta_{\perp}^2$  term in the electric field equation, representing diffraction of the laser beam, loses meaning. Eliminating the spatial dependence of the fields and the diffraction term yields equations homologous to those found in [21].

The resonance frequency of the cavity and the frequency corresponding to the atomic transition are not equal. Therefore there are two non-equal resonance frequencies between which a compromise for a single optical frequency  $\Omega$  must be found. We assume that the space and time coordinates of both complex quantities E and P are separable and that they both evolve at the same frequency. Setting  $E = E_M e^{-i\Omega t}$  and  $P = P_M e^{-i\Omega t}$  in this way, so that  $E_M$  and  $P_M$  are the slowly varying envelopes of the electric field and atomic polarization, is called the rotating wave approximation. Assuming that  $D \approx 1$ , this yields the lasing frequency

$$\Omega = \frac{\gamma_{\perp}\omega_C + k_E\omega_A}{\gamma_{\perp} + k}.$$
(3.3)

In a class B laser, of which a neodymium laser is an example, the decay rate  $\gamma_{\parallel}$  of the atomic polarization P is large enough compared to those of the other two variables that the polarization is essentially determined by the instantaneous values of the electric field and inversion [?].

By setting  $\dot{D} = 0$  in Eqs. (3.1), it is possible to solve for the class B laser equations from the class C laser equations. This process is called "adiabatic elimination," and yields:

$$\dot{E} = -i \left(\omega_c - \Omega\right) E + k_E \left(D - 1\right) E$$
  
$$\dot{D} = \gamma_{\parallel} \left[D_0 - D \left(1 + |E|^2\right)\right]. \qquad (3.4)$$

In this case the spatial dependence has already been dropped, and E(t) represents the slowly varying envelope of the electric field. The electric field oscillations corresponding to the optical frequency  $\Omega$  have been factored out in the rotating wave approximation. ( $\omega_c - \Omega$ ) is the difference between the frequency of the laser light and the resonance frequency of the cavity. Because of this laser "detuning," the electric field is not exactly in phase with itself after one round trip through the cavity. Differences in the optical frequencies are especially important when coupling two lasers together via optical injection.

If such effects are not important, then it is possible to further transform the equations to represent the intensity  $I = E^*E$  and the population inversion D, such that they read,

$$\dot{I} = -k_0 I + g I D,$$

$$\dot{D} = -\gamma D - 2gID + \gamma D_0. \tag{3.5}$$

Here and for the rest of this dissertation we have dropped the subscript  $\parallel$  from the decay rate  $\gamma$  of the population inversion. The decay rate  $k_0$  of the intensity is twice the decay rate  $k_E$  of the electric field. Intensity and population inversion here are both in units of [unit power] per [unit area]. g = 1/2 [unit area]×[unit energy]<sup>-1</sup> is the field-matter coupling constant. (The factor of 1/2 takes into account the conversion from electric field to intensity.)

Neodymium lasers, such as the one used here and modeled by Eqs. (3.4) and (3.5), are often employed for research and industrial applications in either stable (c.w.) or regularly pulsed modes of operation [16].

#### 3.3 Single-Mode Laser Model with Feedback

There are many ways to generate chaotic waveforms from lasers for uses such as transmitting digital information [61]. A simple way to destabilize a class-B laser without an external drive is through the use of feedback. This configuration was proposed for the first time in 1986 for a  $CO_2$  laser with an intracavity electro-optic modulator (EOM) [29].

Here, we first model a single-mode Nd:YAG laser with feedback by a set of three coupled differential equations, one each for the laser intensity, the population inversion, and the voltage of the feedback loop.

Let us consider the experimental apparatus shown in Fig. 2.5 on page 19. The simplest way to describe such a system is by three first-order differential equations, one each for the laser intensity I, the population inversion D, and the modulation voltage V applied to the intracavity AOM:

$$\frac{dI}{dt} = -k_0 \left[ 1 + a \sin^2 \left( \frac{\pi V}{V_{mod}} \right) \right] I + gID,$$

$$\frac{dD}{dt} = -\gamma D - 2gID + \gamma D_0,$$

$$\frac{dV}{dt} = -\beta \left( V - \tilde{B} - \tilde{f}I \right).$$
(3.6)

The cavity loss parameter  $k_0$  is the loss of intensity within the cavity per round trip, not accounting for the dynamically variable loss induced by the AOM. The parameter *a* is the modulation strength of the additional losses introduced by the AOM modulation signal,  $\gamma$  is the population inversion decay rate,  $\beta$  is the damping rate of the feedback loop ( $2\pi$  times the cut-off frequency), *g* is the field-matter coupling constant,  $\tilde{f}$  is the scaling between intensity incident on the photodiode and the voltage read from it times the amplification of the differential amplifier, and  $\tilde{B}$ is a bias voltage applied to the modulator preamplifier. The pump parameter,  $D_0$ , represents the population inversion induced by the action of the diode pump beam.

With suitable normalization,

$$P = \frac{D_0 g}{k_0},$$
  

$$f = \frac{\pi \tilde{f} \gamma}{(2gV_{mod})},$$
  

$$B = \frac{\pi \tilde{B}}{V_{mod}},$$
  

$$x = \frac{2gI}{\gamma},$$
  

$$y = \frac{gD}{k_0}, and$$
  

$$z = \frac{\pi V}{V_{mod}},$$
  
(3.7)

these equations become

$$\frac{dx}{dt} = -k_0 x \left[ 1 + a \sin^2 (z) - y \right],$$

$$\frac{dy}{dt} = -\gamma \left( y - P + xy \right),$$

$$\frac{dz}{dt} = -\beta \left( z - B - fx \right).$$
(3.8)

In this notation, the intensity x is normalized to the saturation intensity, the population inversion y and the pump P are normalized to the threshold population inversion, and the feedback z and bias B are normalized relative to the range of the modulation. For our model, we use the parameter values  $k_0 = 6.6 \times 10^7 \text{ s}^{-1}$ ,  $\gamma = 4.166 \times 10^3 \text{s}^{-1}$ ,  $\beta = 6.28 \times 10^5 \text{s}^{-1}$ , a = 0.052, and P = 1.85. f can be tuned between zero and one.

Some of these parameters ( $\gamma$  and  $\beta$ ) are physical constants already known. It is possible to estimate the other parameters by observing the dynamics of the laser or feedback loop. For example, P is merely the pump power of the diode laser divided by the threshold pump power.

# 3.4 Steady-State Solution and Relaxation Oscillations

From Eqs. (3.8), we can show that the condition

$$\arcsin\left(\sqrt{\frac{P}{a\left(1+\bar{x}\right)}-\frac{1}{a}}\right)-\bar{x}f-B=0\tag{3.9}$$

is fulfilled by any stationary solution of the model. Once the solutions of Eq. (3.9) are known, the stationary values of the other two variables are,

$$\bar{y} = \frac{P}{1+\bar{x}}, 
\bar{z} = B + \bar{x}f.$$
(3.10)

When the laser without feedback is perturbed away from equilibrium, first-order perturbation analysis shows that it will display relaxation oscillations at the frequency given by

$$F_{Rel} = \frac{\sqrt{\gamma k_0 \left(P - 1\right)}}{2\pi}.$$
 (3.11)

The inverse of this frequency can be considered a fundamental characteristic time scale of the laser dynamics. In the case of the dynamics studied here, the laser relaxation oscillation frequency ranges from 50 to 80 kHz, corresponding to the range of 12.5 to 20.0 ms.

It is possible to estimate the decay rate  $k_0$  of the laser intensity from Eq. (3.11) and plot of  $F_{Rel}^2$  vs. the pump power, such as in Fig. 2.8 on page 2.8. In the case of the data for Fig. 2.8, the cavity loss per round trip  $q \approx 0.032$ .  $k_0 = cq/(2L)$  where c is the speed of light and L = 0.10 m was the optical length of the cavity at that time. This gives us an estimate for  $k_0$  of  $4.7 \times 10^7$  s<sup>-1</sup>. In Chapter 4, the value  $k_0 = 6.6 \times 10^7$  s<sup>-1</sup> will be used as stated above. The model presented here as Eq. (3.8) is used successfully to understand the relaxation oscillations presented by the laser as well as to understand the range of stability of the steady state solution in Chapter 4.

To adequately reproduce the more complicated dynamics of the laser, it is necessary to account for the fact that more than one laser mode is present within the cavity.

#### 3.5 Multimode Laser Model

The Tang-Statz-deMars (TSD) two-mode model will be used in Chapter 6 and be shown to offer an improved simulation of the laser chaos itself over the single mode model. In addition, the TSD two-mode model does a good job of simulating the unexpected phenomena presented in that chapter.

The TSD model was proposed in Ref. [55] and is used in Refs. [56, 57, 58, 59]. The original model has five equations; we add a sixth for the feedback loop voltage. The system of equations representing the laser with feedback is,

$$\frac{dn_0}{dt} = w_0 - n_0 - \gamma_1 (n_0 - n_1/2) s_1 - \gamma_2 (n_0 - n_2/2) s_2,$$

$$\frac{dn_1}{dt} = n_0 \gamma_1 s_1 - n_1 (1 + \gamma_1 s_1 + \gamma_2 s_2),$$

$$\frac{dn_2}{dt} = n_0 \gamma_2 s_2 - n_2 (1 + \gamma_1 s_1 + \gamma_2 s_2),$$

$$\frac{ds_1}{dt} = K \left[ \gamma_1 (n_0 - n_1/2) - 1 - a \sin^2 (z) \right] s_1,$$

$$\frac{ds_2}{dt} = K \left[ \gamma_2 (n_0 - n_2/2) - 1 - a \sin^2 (z) \right] s_2,$$

$$\frac{dz}{dt} = -\beta \left[ z - B - f (s_1 + s_2) \right].$$
(3.12)

where  $n_0$  is the space-averaged component of population inversion density with spatial hole burning normalized by the threshold value.  $n_1$  and  $n_2$  are the first spatial Fourier components of population inversion density for the two longitudinal modes.  $s_1$  and  $s_2$  are the normalized amplitudes of the lasing intensities for the two modes.  $w_0 = 1.1$  is the optical pump parameter scaled to the laser threshold.  $\gamma_1=1.00$  and  $\gamma_2=0.98$  are the gain coefficients for the two modes.  $K = \tau/\tau_p=36840$ , where  $\tau$  is the upper state lifetime and  $\tau_p$  is the photon lifetime in the laser cavity.

The sixth equation, for the voltage z applied to the AOM driver, is added to the normal TSD model.  $\beta$ =144 is the cutoff frequency of the filter (corresponding to 100 kHz). The bias of the feedback loop is represented by B, a = 0.015 is the amplitude of the loss modulation, and the amplitude of the photodiode response is f = 0.75. Time for the equations is scaled by  $\tau$ .

It might quickly be noticed that there are three equations for the population inversion  $(n_0, n_1, \text{ and } n_2)$  although there are only two cavity modes. Three equations are necessary to account for an effect called "spatial hole burning." As depicted in Fig. 2.2 on page 13, the electric field densities of the two modes are localized differently within the laser crystal. The two modes therefore have access to slightly different population inversions  $(n_0 - n_1/2)$  for mode 1 and  $(n_0 - n_2/2)$  for mode 2.

Further information about the characteristics and applications of this model is also available in Ref. [60].

In the following chapter, we will proceed with experimental investigation of the laser dynamics.

### Chapter 4

# **Chaotic Function Generation**

#### 4.1 Motivation

There is continued growth in the field of nonlinear dynamics and an ever-increasing appreciation of its applications among researchers. With this growth, devices capable of generating complicated waveforms may be of use to complement the standard function generator. In this chapter, we use the laser system as an example of such a device and present the concept of its operation.

Chaotic waveforms from lasers have been shown to be useful for such purposes as transmitting digital information [61, 62]. It may also be possible to generate waveforms appropriate for encoding speech.

Within a range of tuning, the laser described in Chapter 2 exhibits a variety of dynamics, including low-dimensional chaos. As chaotic waveforms are not periodic functions, these dynamics are best distinguished from one another by their statistical characteristics. It is possible to create a table of the recorded statistical characteristics of the chaotic time series for different values of the tuning parameter(s). This table would be, or would at least resemble, a bifurcation diagram. The idea is merely to use the bifurcation diagram backwards, instead of trying to use it only to describe how the system behaves, we would like to know where it behaves the most similarly to some desired behavior.

Once such a lookup table is built, the laser may be used to generate waveforms

selected by the information recorded about them. In this chapter will calculate and record the discrete probability density functions, the power spectra, and the leading Lyapunov exponents of the waveforms generated by the laser for different values of the feedback loop bias B.

# 4.2 Laser Dynamics and Control

The dynamics of the Nd:YAG laser with feedback reveals the presence of two distinct regions of instability easily accessible by varying the bias voltage in the feedback loop. These regimes bound the domain where the stationary solution is stable. Toward positive bias, the transition to oscillatory behavior occurs through a subcritical Hopf bifurcation. Toward negative bias, the transition occurs through a supercritical Hopf bifurcation. Though the sub- and supercritical bifurcations cannot be distinguished in the linearized model, the presence of hysteresis [3] in the numerical simulation and in the laser system are good indications that the bifurcation at positive bias is subcritical.

As the transition to chaos in the experiment is very abrupt and leads quickly to high-amplitude chaos, we focus part of our attention on softening this transition to increase the variety of lower-amplitude chaotic dynamics available to the function generator. Control of chaos often takes advantage of steady states or periodic orbits inside a chaotic attractor to regularize dynamics. One of the major ways in which this is accomplished is by applying perturbations to state variables or system parameters. The key idea in the pioneering work by Ott, Grebogi, and Yorke (OGY) [64] is to use linear control theory and feedback on a system parameter to direct the motion of trajectories along the stable manifold of an unstable state. A multivariate control method, occasional proportional feedback [65], and some variations of it have been successfully applied to stabilize unstable steady states and periodic orbits in a multimode Nd:YAG frequency-doubled laser (the green problem) [66, 67, 68, 69]. This control method has been shown to be distinct from the OGY method in Ref. [70]. The problem of the stabilization of an unstable steady state can also be approached by using a derivative control on a state variable, i.e., feedback control loops containing terms proportional to the derivative of the output of a given system. Derivative control has been successfully applied to many systems, for example [71, 72, 73, 74, 75], for stabilization. Other forms of control have also been used to maintain chaos, as in [76].

As we would prefer to use control for a purpose intermediate between suppressing and sustaining chaos, we look to employ the derivative control to alter the dynamics within the chaotic regions without eliminating the chaos.

The chapter is organized as follows. Section 4.3 describes the design of the function generator. In Sec. 4.4, we report experimental measurements of the laser dynamics with feedback as well as with control. We compare the probability distributions, power spectra, and leading Lyapunov exponents of the intensity waveforms the laser generates for different values of the feedback bias. Section 4.5 contains the theoretical model for the laser, and describes the results of linear stability analysis. The results are discussed in Sec. 4.6.

### 4.3 Chaotic Function Generator Setup

In addition to the basic laser and feedback loop setup described in Chapter 2, we add a second branch of the feedback network, which we refer to as the control loop. This loop consists of a high-pass filter with a 48 dB/octave rolloff and an amplifier with adjustable gain G; the output of this amplifier is applied to the inverting input of the differential amplifier in Fig. 2.5 on page 19. This differential amplifier has a bandwidth  $\beta$  of 100 kHz. The net gain of the feedback loop is f and the net gain of the control loop is Gf.

The laser system described, with the additions described above, is displayed in Fig. 4.1.

As noted in Chapter 2, the delay of the feedback loop comes in two parts, propagation of the sound wave inside the AOM and transmission time of the electronic



Figure 4.1: Experimental setup of the chaotic function generator system. The solid line connecting the detector to the AOM indicates the main feedback loop generating the dynamics. The dashed line indicates the additional control loop including a reshaping filter, Wavetek model 452, in the high-pass configuration with a slope of 48 dB/octave. With or without the control loop active, the bias B is a tuning parameter that allows access to a variety of laser dynamics.

signal. Transmission of the electronic signal through the coaxial cables introduces a delay on the order of tens of nanoseconds. The speed of sound within the AOM (a PbMoO<sub>4</sub> crystal) is 3630 m/s and the distance between the transducer and the laser beam is approximately 5 mm. This corresponds to a time delay of 1.37 ms.

The effect of the time delay on the dynamics is uncertain. However, we do not observe a peak in the power spectrum corresponding to the delay, which indicates that the effect is probably small. This is further confirmed by the fact that the characteristic frequency associated with the delay time is higher than the cutoff frequency of the feedback loop (f = 100 kHz). The resulting dynamics occurs on the characteristic time of the laser relaxation oscillation frequency (40-80 kHz, see Sect. 4.5) is about ten time longer than the delay time.

#### 4.4 Experimental Results

Figure 4.2 shows short segments of experimental time traces seen by adjusting only the bias of the feedback loop without the effect of the second branch (control loop) shown in Fig. 4.2. These sixteen values of the bias were chosen to illustrate a variety of intensity dynamics observed in the laser system. These dynamics include multiplespike bursting at large positive bias, moderate-amplitude chaos at small negative and small positive bias, and high-amplitude chaotic spiking at large negative bias.

Figure 4.3 shows a bifurcation diagram of the laser dynamics with respect to the bias of the feedback loop, still without the additional control loop present. Each vertical strip in the diagram represents the discrete probability density function (PDF) of the laser intensity at a particular value of bias. The darkness at each location represents the relative amount of time the laser spends at a particular intensity. We have chosen to examine the PDF of the intensity signal because from the PDF we can calculate the standard statistical measures. Arrows at the bottom of the figure indicate locations that have been sampled to create Figs. 4.2, 4.4, and 4.5.

In Fig. 4.4, we have plotted the normalized discrete probability density func-



Figure 4.2: Experimental time traces of the laser intensity with feedback but without the control loop for sixteen values of the bias B, showing chaotic spiking (B > -0.0525 V), chaos just after the Hopf bifurcation (B = -0.0525 V), near steadystate (cw) operation (B = -0.0345 V, -0.0180 V, 0.0000 V, 0.0180 V), another variety of chaotic oscillations (B = 0.0345 V, 0.0525 V, 0.0705 V) and bursting (B = 0.0870 V, 0.1050 V, 0.1230 V, 0.1395 V).



Figure 4.3: Bifurcation diagram of the discrete probability distribution function (PDF) of the laser intensity signal for 200 values of feedback bias B without control. Arrows indicate the bias values at which the time traces in Fig. 4.2 were obtained. The grayscale axis represents the log of the probability.



Figure 4.4: PDFs taken from Fig. 4.3 for the same values of bias displayed in Fig. 4.2. A variety of shapes can be seen and the transition between them is also apparent. The area under each curve is normalized to 1 and the curves are displayed on a log-linear scale. Each PDF was constructed from 65,536 data points.

tions for the same values of bias as in Fig. 4.2. These slices of Fig. 4.3 allow us to distinguish between very different sorts of behavior that appear similar in both the estimates of leading Lyapunov exponent and the bifurcation diagram of Fig. 4.3. Note, for example, the cusped tail in the PDF of the spiking behavior ( $B \leq -0.0705$  V) vs. the more exponential tail in the PDF of the bursting ( $B \geq 0.0870$  V).

The power spectra of the dynamics at the same values of bias are displayed in Fig. 4.5. The main peak when the laser is near steady-state operation (B = -0.0345 V, -0.0180 V, 0.000 V) represents the laser relaxation oscillation frequency.

Figure 4.6 shows a bifurcation diagram of the laser dynamics with the control loop active. The control loop significantly alters the dynamics of the laser for all



Figure 4.5: Power spectra for the laser intensity signals of Fig. 4.2. The peaks observed near steady state operation represent the characteristic laser relaxation oscillation frequency (approximately 79 kHz). Away from these values of bias, the laser exhibits a variety of broad spectral shapes. Above 200 kHz, there may be additional peaks weaker than those between 0 and 200 kHz. Those peaks are imposed upon the expected exponential decay of the power as frequency increases.



Figure 4.6: Bifurcation diagram of the PDF of the laser intensity signal for 200 values of bias with the control loop active. The controlling filter rolloff frequency  $\omega = 5$  kHz, and the gain  $\Gamma$  has been tuned to produce the largest window of steady-state operation. Arrows indicate the bias values for which the time traces, PDFs, and power spectra are taken in Figs. 4.7, 4.8, and 4.9. The grayscale axis represents the log of the probability.

regions where the laser was not already in steady state. In the case of this diagram, the onset of chaos is significantly delayed in the direction of negative bias. For positive bias values, the regions of bursting are replaced by limit cycles or chaos, both with smaller amplitudes than in Fig. 4.2.

The effect of our filtering control on the time series is shown in Fig. 4.7, where we display excerpts of the traces of the laser intensity for the same sixteen values of bias as in Fig. 4.2, but with the control loop active. Note the expanded intensity scale.



Figure 4.7: Selected time traces of the laser intensity with control for sixteen values of the bias B, showing chaotic oscillations ( $B \leq -0.1050$  V), near-periodic behavior (B = -0.0870 V, -0.0705 V, 0.0000 V, 0.0180 V, 0.0345 V, 0.0525 V), near steady-state (cw) operation (B = -0.0525 V, -0.0345 V, -0.0180 V), and chaotic spiking ( $B \geq 0.070$  5V).

Table 4.1 compares the leading Lyapunov exponents calculated for the sets of data in Fig. 4.2 where the feedback loop is active and 4.7 where both feedback and control are active. These calculations were performed with the software package cspW2, which uses the algorithms described in [77]. When the intensity signal becomes nearly steady, the ambient noise recorded by the oscilloscope (0.02 peak-to-peak in the arbitrary units used in these figures) is registered by the cspW package as a chaotic signal. For this reason, we never calculate a nonpositive leading Lyapunov exponent.

Table 4.1: Largest Lyapunov exponents as calculated by the cspW program (see Ref. [77] for the time series used to generate all figures in this section. When the laser is near steady state operation, noise in detection equipment results in calculation of what is most likely a false positive Lyapunov exponent. Thus the larger values for -0.0525 < B < 0.0180 are presumably false positive Lyapunov exponents and the positive Lyapunov exponents calculated for the other values of bias are more likely indications of chaos.

|            | Largest Lyapunov Exponent $(10^6 \text{ s}^{-1})$ |         |
|------------|---|---------|
| Bias $(V)$ | Feedback  | Control |
| -0.1230    | 2.12  | 0.38    |
| -0.1050    | 2.06  | 0.50    |
| -0.0870    | 2.00  | 0.82    |
| -0.0705    | 1.70  | 1.06    |
| -0.0525    | 0.28  | 1.92    |
| -0.0345    | 2.34  | 2.10    |
| -0.0180    | 2.88  | 2.36    |
| 0.0000     | 1.66  | 1.08    |
| 0.0180     | 0.88  | 0.52    |
| 0.0345     | 0.34  | 0.30    |
| 0.0525     | 0.96  | 0.24    |
| 0.0705     | 0.78  | 0.14    |
| 0.0870     | 2.30  | 0.20    |
| 0.1050     | 1.34  | 0.16    |
| 0.1230     | 1.86  | 0.14    |
| 0.1395     | 1.80  | 0.28    |



Figure 4.8: PDFs for the same values of bias as in Fig. 4.7. A variety of shapes distinct from those in Fig. 4.4 are evident for large negative and positive bias values.

In Fig. 4.8, we present the PDFs calculated for these the waveforms produced with control at the same values of bias. The intensity scale has been expanded because the range of the laser intensity is smaller. Figure 4.9 displays the corresponding power spectra. Both PDFs and power spectra are markedly different than those found in Figs. 4.4 and 4.5.

The form of control explored in this section renders the transition from steady state to chaos more gradual. The control loop we have used to quench the dynamical range of the laser intensity has increased the variety distinct behaviors that the laser intensity signal can exhibit.

In this section, we have been able to evoke a wide variety of chaotic waveforms from the laser. These waveforms can be distinguished from one another by their statistical characteristics and power spectra. In this sense, we have been able to



Figure 4.9: Power spectra for the laser intensity signals shown in Fig. 4.7. The peaks observed near steady state operation again represent the laser relaxation oscillation frequency. Away from these values of bias, the spectral characteristics of the laser signal differ significantly from those in Fig. 4.4

record and classify the laser intensity dynamics. One can now view the laser as a device for selecting and generating chaotic waveforms with desired characteristics.

## 4.5 Numerical Results

We now investigate how analysis of this system using the single-mode laser model with feedback presented in Chapter 3 as Eqs. (3.8) on page 27 (reprinted below for convenience) reproduces the most basic laser dynamics, including steady state, relaxation oscillations, and the transition to chaos at negative bias and the periodic spiking at large negative bias. We can also use this model to lend some insight into the effect of control loop.

$$\frac{dx}{dt} = -k_0 x \left[ 1 + a \sin^2 (z) - y \right],$$

$$\frac{dy}{dt} = -\gamma \left( y - P + xy \right),$$

$$\frac{dz}{dt} = -\beta \left( z - B - fx \right).$$
(4.1)

The parameter values used in this section are a = 0.052,  $k_0 = 6.6 \times 10^7 \text{ s}^{-1}$ ,  $g = 4.166 \times 10^4 \text{ s}^{-1}$ , P = 1.85,  $\beta = 6.28 \times 10^5 \text{ rad/s}$ , and f = 0.75.

In order to determine the stability of an equilibrium solution  $\bar{\mathbf{r}}(x, y, z)$  of Eqs. (3.8), we consider the Jacobian Matrix  $\mathbf{D}f(\mathbf{r})$  of the partial derivatives evaluated at  $\bar{\mathbf{r}}$ . Stability of  $\bar{\mathbf{r}}$  in the face of a small perturbation is determined by the eigenvalues of the linearized flow represented by  $\dot{\mathbf{v}} = \mathbf{D}f(\bar{\mathbf{r}})\mathbf{v}$ , where the vector  $\mathbf{v}$  denotes a small deviation from the fixed point. The real parts of the three eigenvalues are reported in Fig. 10. Note that the leading eigenvalue, where positive, is of the order of  $10^6 \, \mathrm{s}^{-1}$  as are the largest Lyapunov exponents calculated for the experiment in Table 4.1. The values of the bias B where the fixed point is unstable occur where the real part of any eigenvalue is positive. In our case, the fixed point undergoes a Hopf bifurcation where an eigenvalue intersects the zero line in Fig. 4.10. The



Figure 4.10: Plots of the real parts of the eigenvalues of the Jacobian matrix associated with Eqs. (3.8) as a function of the tuning parameter B. The parameter values are a = 0.052,  $k_0 = 6.6 \times 10^7 \text{ s}^{-1}$ ,  $g = 4.166 \times 10^4 \text{ s}^{-1}$ , P = 1.85,  $\beta = 6.28 \times 10^5$ rad/s, and f = 0.75. The leading eigenvalue, where positive, is of the order of  $10^6$ s<sup>-1</sup>, comparable to the Lyapunov exponents calculated in Table 4.1

imaginary part of this eigenvalue corresponds to a frequency of 76 kHz, comparable with the relaxation oscillation frequency of 76.9 kHz given by Eq. (3.11) on page 28.

The bifurcation diagram for the model system is shown in Fig. 4.11. The model parameter B, corresponding to the feedback bias, is slowly increased from negative to positive in a time of 0.05 s, considerably longer than the microsecond time scale of the laser dynamics. On the vertical axis, the local maxima and minima of the laser intensity are shown. Toward negative bias, there is evidence of a cascade of subharmonic bifurcations ending with chaos after a supercritical Hopf bifurcation. However, the model does not display bursting for positive bias and the spiking at large negative bias becomes regular in amplitude rather than chaotic as



Figure 4.11: Bifurcation diagram of the local maxima and minima of the laser intensity vs. B. The parameter values are the same as those in Fig. 4.10. The scan time is 0.05 s, and the steady state value is displayed where it is stable. The period doubling cascade begins around B = -0.64. Note the logarithmic scaling of the xaxis.

in the experimental system. The model is also much more sensitive to the choice of parameters P, a, and f than the laser system.

For positive values of B, the transition to a region of large amplitude oscillations occurs through a subcritical Hopf bifurcation (B = 1.05). This bifurcation exhibits hysteresis as the control parameter is varied in the opposite direction (B = 0.89). The model does not display the chaotic bursting evident in the experiment.

If we denote the input signal, proportional to the laser intensity, by x(t), the output signal u(t) of an RC first-order filter is given by the differential equation  $\dot{u} = -\omega u + \dot{x}$ , where  $\omega$  is defined as  $2\pi$  times the rolloff frequency ( $\omega = 1/RC$ ). In

our model, we insert this perturbation into the feedback loop by adding a term to the equation governing z(t). With a gain factor  $\Gamma$ , the perturbation signal affects the dynamics in following way

$$\dot{x} = -k_0 x \left[ 1 + a \sin^2 (z) - y \right],$$
  

$$\dot{y} = -\gamma \left( y - P + xy \right),$$
  

$$\dot{z} = -\beta \left[ z - B - f \left( x - \Gamma u \right) \right],$$
  

$$\dot{u} = \dot{x} - \omega u.$$
(4.2)

It is always possible to find a region in the parameter space  $(\omega, \Gamma)$  where the control in Eqs. (4.2) stabilizes the steady state for a given value of B. But a value of  $\omega$  near to or lower than the leading frequency component of the laser intensity fluctuations may significantly alter the system dynamics without stabilizing the steady state. In our case, this leading frequency corresponds to the frequency of the limit cycle just after the Hopf bifurcation.

In the experiment, we use a higher-order filter to render the process more selective in frequency. Such a selective filter control can be modeled by cascading several first-order filters. Here, we limit the analysis to a third-order filter. The overall dynamics is now described by the following system of differential equations

$$\dot{x} = -k_0 x \left[ 1 + a \sin^2 (z) - y \right], 
\dot{y} = -\gamma (y - P + xy), 
\dot{z} = -\beta \left[ z - B - f (x - \Gamma u_3) \right], 
\dot{u}_1 = \dot{x} - \omega u_1, 
\dot{u}_2 = \dot{u}_1 - \omega u_2, 
\dot{u}_3 = \dot{u}_2 - \omega u_3.$$
(4.3)

The results of the linear stability analysis for both model systems from Eqs. (4.2) and Eqs. (4.3) are reported in Figs. 4.12 (a) and (b) respectively, with the most neg-



Figure 4.12: a) Real parts of the eigenvalues of the controlled system of Eqs. (4.2) (first-order filter) vs. B. b) Real parts of the eigenvalues of the controlled system of Eqs. (4.3) (third-order filter) vs. B. The parameters of the control loop are  $\omega = 5$  kHz and  $\Gamma = 0.88$ . The magnitude of the leading eigenvalue is reduced in both cases by a factor of 10 from that calculated for the model with feedback only.

ative exponent not displayed in both cases. Both first- and third-order filters reduce the range of the magnitude of the leading eigenvalue.

The global effect of the controlling perturbation on the simulated laser dynamics, as seen in the bifurcation diagram in Fig. 4.13, shows the enlargement of the stability domain of the controlled dynamics. The chaotic region after the supercritical Hopf bifurcation is replaced by the stationary solution. The subcritical Hopf bifurcation at positive B is now replaced by a supercritical Hopf bifurcation.

Comparison of the power spectra and time-delay embeddings for theory and experimental results are displayed in Fig. 4.14. We present the chaotic attractor just after the onset of chaos (after the Hopf bifurcation) to compare it with the model. We have chosen model parameters inside the small chaotic window. The leading



Figure 4.13: Bifurcation diagram of the local maxima and minima of the laser intensity x vs. B for the controlled dynamics of Eqs. (4.3). The laser model parameter values are the same as those in Fig. 4.11. The scan time is 0.05 s, and again the steady state value is displayed where it is stable. The parameters of the control loop are  $\omega = 5$  kHz and g = 0.88. The supercritical bifurcation at negative B now begins near B = -0.96.



Figure 4.14: Left side, power spectra and attractor reconstructed from the intensity series of the experiment with feedback only at a bias of -0.045 V and with embedding time delay t = 0.8 ms. Right side, power spectra and attractor reconstructed from the x values of the simulation of Eqs. (3.8) (also with feedback only) at B = -0.711 with embedding time delay t = 0.8 ms.

frequency in the numerical simulation is roughly 75 kHz compared to about 83 kHz in the experiment. Both peaks are near the relaxation oscillation frequency of the laser and, as a consequence, the Hopf frequency. In both spectra, subharmonic peaks are observed at approximately one-third the frequency of the main peak.

The single mode laser model fails to account for the chaotic laser dynamics throughout a wide range of B values. The single mode model does account for the steady state value, the relaxation oscillation frequency, and the periodic spiking orbit for large negative bias values. In Chapter 6 the multimode model will be used with greater success to model the chaotic laser dynamics.
### 4.6 Chaotic Function Generator Discussion

Until recently, typical laser applications have focused on either steady state operation or regularly pulsed operation. In this chapter, the dynamics of an Nd:YAG laser subject to feedback modulation of the intracavity losses via an AOM have been explored. Tuning the bias of the feedback loop allows easy exploration of a range of intensity dynamics that includes a sudden transition from steady state to highamplitude spiking and bursting. An additional branch of the feedback loop that suppresses higher-frequency components of the feedback alters the dynamics of the laser while preserving the chaos. With this controlling loop active, the transition to chaos is more gradual and low-amplitude chaotic oscillations are observed.

We have demonstrated that a laser with suitably designed feedback can serve as a tunable generator of chaotic "functions." These waveforms can be characterized through bifurcation diagrams that record discrete probability density functions, power spectra, or other measures of the dynamics. Bifurcation diagrams can then be used as libraries or lookup tables to select particular chaotic signals by their statistical characteristics.

In the next chapter, we will study phase synchronization of the chaotic laser dynamics to an external drive.

### Chapter 5

### **Phase Synchronization Dynamics**

### 5.1 Motivation

A chaotic system often responds in a complicated way to external driving signals. One such response is phase synchronization, where the amplitudes of response and drive may be poorly correlated, but a relationship becomes evident if suitable phases are defined [6, 7, 8].

Phase synchronization of a chaotic system to a single external periodic drive [6, 7, 45, 78] as well as to another chaotic system [8, 79, 46] has been observed in numerical models and experiments. Phase synchronization of chaos is also known to be important in many biophysical phenomena, including brain function [80], kidney function [81], and cardio-respiratory synchronization [82].

In a recent paper [51], Breban and Ott addressed the natural question of how phase synchronization manifests itself when a chaotically behaving system is driven by two regular signals of different periods. The phenomena they found include synchronization to one or the other of the driving frequencies, synchronization to the average frequency, and competition (alternating locking and slipping) between these.

In this paper, we report investigations of phase synchronization in a chaotic laser driven by two sinusoidal signals. We think of this laser as a generic, lowdimensional chaotic system. We observe and quantify competition between the two



Figure 5.1: Experimental setup of a diode-pumped Nd:YAG laser with an intracavity AOM. The solid line connecting the detector to the AOM indicates the feedback loop generating the chaotic dynamics. A signal s(t) from the arbitrary function generator may be added to the voltage in the feedback loop.

frequencies and their average for synchronization of the laser. Our results give experimental realizations of the phenomena predicted in Ref. [51].

### 5.2 External Driving Setup

Our system is the laser with feedback shown in Fig. 5.1. The laser consists of a diode laser pump source, a Neodymium-doped Yttrium Aluminum Garnet (Nd:YAG) laser crystal, an intracavity acousto-optic modulator (AOM), and an output coupler. The laser intensity is detected by a photodiode and the photodiode voltage is applied through a feedback loop to the AOM. The laser operates in a  $TEM_{00}$  Gaussian transverse mode but is free to lase in multiple longitudinal modes.

The AOM modulates the cavity loss around its median value  $k_0$ . The cavity

loss induced by the modulator is proportional to  $\sin^2(\pi V/V_{mod})$  where V is the voltage applied to the modulation input of the AOM driver and  $V_{mod} = 0.5$  V is the saturation voltage of the driver. The laser is pumped at a ratio of pump power to laser threshold of approximately 1.9.

The gain  $\Gamma$  and bias B of the feedback loop can be used to select among a variety of laser dynamics [83]. We tune  $\Gamma$  and B to a region where the intensity displays chaotic spiking with both irregular spike heights and irregular inter-spike intervals.

The feedback loop also includes a differential amplifier, with bandwidth  $\beta =$  100 kHz. The output of an arbitrary function generator (40 Msamples/s, 12 bits output resolution) can be added to the feedback signal through this amplifier.

Measurements of the photodiode signal are made with a digital oscilloscope with 14 bits of precision. We choose to perform measurements at 5 Msamples/s to allow for a well-resolved construction of the Hilbert phase.

#### 5.3 Hilbert Phase for Chaotic Signals

Figure 5.2 (a) shows a sample of the chaotic intensity dynamics of the laser with feedback but in the absence of modulation. Among the many options, we turn to the analytic signal to construct a phase for the irregular oscillations [8].

For any real time series  $V^{(r)}(t)$ , such as the laser intensity, we can compute the corresponding analytic signal  $V(t) = V^{(r)}(t) + iV^{(i)}(t)$ , where

$$H\left[V^{(r)}(t)\right] = V^{(i)} \equiv \pi^{-1} P \int_{-\infty}^{\infty} V^{(r)}(t')(t'-t)^{-1} dt'$$
(5.1)

is the Hilbert transform of  $V^{(r)}(t)$  and P denotes taking the principal value of the integral. Writing  $V(t) = A(t)e^{i\phi_H(t)}$ , where A(t) is a real function, we obtain the Hilbert phase  $\phi_H(t)$  of the real signal  $V^{(r)}(t)$ . This phase describes changes in the field envelope and can be evaluated from experimental time series using the fast Fourier transform [84].

Figure 5.2 (b) shows the Hilbert phase constructed from the signal shown in



Figure 5.2: a) Time series recorded from the laser with feedback but no modulation. The small-scale wiggles in the time series are artifacts of the rendering. b) The reconstructed phase of the signal from (a), modulo  $2\pi$ . c) Power spectrum of the laser. The spectrum is broad and its highest peak is at approximately 64 kHz. We can also perceive what appear to be 1:2 and 1:3 subharmonic peaks at 32 kHz and 21 kHz respectively.

Fig. 5.2 (a). The constructed phase  $\phi_H$  corresponds in a sensible manner with the recorded dynamics of the laser intensity.

### 5.4 Phase Synchronization

The power spectrum of the laser in the same conditions is displayed in Fig. 5.2 (c). The main peak in the power spectrum (approximately 64 kHz) is close to the relaxation oscillation frequency  $f_{rel}$  of the laser dynamics.

Beyond the range of frequency displayed, the power spectrum of the recorded laser intensity signal decays exponentially with respect to frequency down to the 14 bit resolution of the digital oscilloscope.

We now add a single-frequency sinusoidal perturbation onto the voltage feedback signal. For sufficient amplitude and for choices of f close to the relaxation oscillation frequency, the phase of the chaotic signal typically synchronizes intermittently to the perturbation. The laser intensity remains chaotic unless the perturbation becomes large compared to the feedback signal (approximately 90 mV peak-to-peak), where the intensity dynamics consists of regular spikes fully entrained to the driving frequency.

We consider the constructed phase  $\phi_H(t)$  of the chaotic laser intensity signal as unbounded (not taken modulo  $2\pi$ ) and define  $\Delta \phi$  for the chaotic signal with respect to a given frequency f as

$$\Delta\phi(t) = \phi_H - 2\pi f t. \tag{5.2}$$

In Fig. 5.3 we display the phase difference  $\Delta \phi$  between the sinusoidal modulation and the chaotic signal for four values of the driving frequency f. When f is below the laser relaxation oscillation frequency  $f_{rel}$ , phase slips occur predominantly with the phase of the laser advancing more quickly than that of the driving (slanted segments of  $\Delta \phi$  with positive mean slope). When f is above  $f_{rel}$ , phase slips tend to occur in the other direction. Nearly horizontal segments of  $\Delta \phi$  represent times during which the laser has become phase synchronized to the sinusoidal modulation. With similar amplitudes, driving frequencies closer to  $f_{rel}$  tend to entrain the laser



Figure 5.3:  $\Delta \phi$  from the experimental system for four values of a single driving frequency. The AOM is driven with an amplitude of 50mV. In the four curves, we can see long plateaus of synchronization as well as large phase slips. Curve A: f =50.0 kHz, Curve B: f = 60.0 kHz, Curve C: f = 67.8 kHz, Curve D: f = 74.9 kHz.

phase for longer periods of time than frequencies further from  $f_{rel}$ .

The phases of the chaotic signals in all four curves synchronize to the perturbation for intervals as long as several tenths of a ms, or many times the characteristic period of the laser relaxation oscillations. During these intervals of synchronization,  $\Delta \phi$  wiggles irregularly due to the varying shape of the intensity oscillations but remains within a range of width  $2\pi$ .

# 5.5 Two Frequencies Compete for Phase Synchronization

When we introduce driving composed of sinusoids at two frequencies, we are able to observe competition for synchronization between each of the frequencies  $(f_1, f_2)$ and their average  $(f_{AV})$ .

For periods of time containing tens or hundreds of oscillations of intensity, the laser can phase synchronize to one frequency, say  $f_i$ . During this time, the line representing  $\Delta \phi_i = \phi_H - 2\pi f_i t$  will look flat if one ignores fluctuations of magnitude less than  $2\pi$ . At the same time, the lines representing the phase differences between the chaotic laser intensity signal and the other two frequencies will slant down or up, depending upon whether each frequency is greater or less than the frequency that is successfully entraining the chaotic dynamics.

Figure 5.4 shows a plot of the phase differences for  $f_1$ ,  $f_2$ , and  $f_{AV}$  in an experimental run with two-frequency modulation. The two driving frequencies in this case are slightly lower than the relaxation oscillation frequency, and the amplitude of the perturbation is chosen so that entrainment is observed while chaos is preserved. Each inset enlarges a segment during which the phase of the laser signal is entrained dominantly to one of  $f_1$ ,  $f_2$ , or  $f_{AV}$ . Surprisingly, the laser predominantly slips back and forth between synchronization to  $f_2$  and  $f_{AV}$ , even though  $f_1$  is closer to  $f_{rel}$ , see Fig. 5.5.

Following Ref. [51], we plot  $\Delta \phi_2$  versus  $\Delta \phi_1$  from the data in Fig. 5.4 as curve



Figure 5.4:  $\Delta \phi$  for the experiment with two driving frequencies  $f_1 = 62.5$  kHz and  $f_2 = 59$  kHz, both with an amplitude of 50 mV. The top curve represents  $\Delta \phi_2$ , which tends increase with time, the bottom curve represents  $\Delta \phi_1$ , which tends to decrease with time, and the middle curve represents  $\Delta \phi_{AV}$ . The inset boxes display enlarged sections of the three  $\Delta \phi$  curves, each over a time of 0.5 ms.

| Curve | $f_1$ (kHz) | $f_2$ (kHz) | to $f_1$ | to $f_2$ | to $f_{AV}$ | slipping |
|-------|-------------|-------------|----------|----------|-------------|----------|
| А     | 62.5        | 59.0        | 0.06     | 0.34     | 0.21        | 0.39     |
| В     | 59.0        | 57.5        | 0.01     | 0.08     | 0.68        | 0.22     |
| С     | 67.9        | 66.0        | 0.12     | 0.31     | 0.27        | 0.30     |

Table 5.1: The portion of time each of the three plots of  $\Delta \phi_2$  vs.  $\Delta \phi_1$  displayed in Fig. 5.5 phase locks to one of  $f_1$ ,  $f_2$ , or  $f_{AV}$ .

A in Fig. 5.5. The staircase-like structure we observe shows that  $\Delta \phi_1$  is typically locked (within  $2\pi$ ) while  $\Delta \phi_2$  slips (vertical segments), and  $\Delta \phi_2$  is typically locked while  $\Delta \phi_1$  slips (horizontal segments). Line segments with slope close to +1 indicate that the phase of the chaotic signal is slipping with respect to both of the driving frequencies. As  $f_1$  and  $f_2$  are unequal, the slope cannot be exactly +1.

We create a piecewise linear simplification of each  $\Delta \phi$  curve by inspection. The simplification is composed of line segments of zero slope with slanted segments connecting them. The total length of line segment of slope zero within a particular  $\Delta \phi_i$  curve is counted as synchronization to  $f_i$ . The portion of time the laser spends phase synchronized to each frequency has been calculated this way and entered in Table 5.1.

Curves B and C in Fig. 5.5 show similar behaviors typically seen for driving at frequencies above, below, or straddling the laser relaxation oscillation frequency  $f_{rel}$  (approximately 64 kHz). In curve B, climbing bottom to top, synchronization to the average is most prevalent and seems to occur repeatedly for short stretches where  $\Delta \phi_1$  and  $\Delta \phi_2$  also vary by less than  $2\pi$ . These stretches of synchronization to  $f_{AV}$  appear as short sequences with an average slope of -1, as in the inset. In curve C, the driving frequencies are greater than  $f_{rel}$ , also in this case, the net slip in phase is least. In all three cases, the net chaotic phase slip is least with respect to the driving frequency closest to  $f_{rel}$ , even if the chaotic phase locks more often to the other frequency or  $f_{AV}$ . The portion of time that the intensity signal stays phase-locked for all two-frequency data are displayed in Table 5.1.

We can easily distinguish between locking to  $f_1$ ,  $f_2$ , and  $f_{AV}$ , and slipping rel-



Figure 5.5:  $\Delta \phi_2$  vs.  $\Delta \phi_1$  for three experimental time series with different driving. Phase synchronization to  $f_1$  appears as vertical trails, synchronization to  $f_2$  appears as horizontal trails. Curve A: the same 13 ms time series as in Fig. 5.4. The inset shows that during the long time apparently synchronized to  $f_1$ , the phase is often in fact synchronized to  $f_{AV}$  or  $f_2$ . Curve B: A 13 ms time series with  $f_1 = 59$  kHz,  $f_2 = 57.5$  kHz. Synchronization to the average can be seen when  $\Delta \phi_1$  increases while  $\Delta \phi_2$  decreases, as in the inset. Curve C: A 6.5 ms time series where both frequencies are greater than the relaxation oscillation frequency of the laser,  $f_1 = 67.9$  kHz,  $f_2 = 66$  kHz.



Figure 5.6: Three-dimensional time-delay embedding of portions of the time series used to generate Fig. 5.4 and curve A in Fig. 5.5. The time delay is 1  $\mu$ s. The blue, green, and red curves represent segments of the time series during which the phase  $\phi_H$  is synchronized to  $f_1$ ,  $f_{AV}$ , and  $f_2$  respectively. The black curve represents a segment during which the phase is slipping with respect to all three frequencies. It is difficult to distinguish between the three cases of synchronization by examining the attractor.

ative to all of them by examining a plot of  $\Delta \phi_2$  vs.  $\Delta \phi_1$  such as in Fig. 5.5. Could we as easily make this determination in some other way? In Fig. 5.6, we have generated a delay-embedding of the intensity series used to generate Fig. 5.4. We have used 0.1 ms segments of time taken from the three insets. When the laser is synchronized to  $f_1$ ,  $f_{AV}$ , or  $f_2$ , the attractor is colored blue, green, or red, respectively. When the phase of the laser is slipping relative to all three, the curve is black. In longer times, the colored curves map out visually indistinguishable attractors. When the phase is slipping, the intensity tends to spend more time at either the very inside or outside of the attractor. It is not straightforward to determine which frequency the laser phase is synchronized to, or even if it is synchronized at all, from the structure of the attractor.

One might also try to detect phase synchronization by examining the spectrogram. However, discerning synchronization over short times requires a small time window for the calculation of the spectrum while achieving high resolution in frequency requires a longer time window. Therefore it is difficult to observe competition between two frequencies with a spectrogram when the durations of synchronization may be short and the frequencies of interest may be close together.

We have also tried to detect phase synchronization by examining a 3-D timedelay embedding of the attractor. However, we discover that trajectories within the attractors corresponding to synchronization to  $f_1$ ,  $f_{AV}$ , or  $f_2$  are visually indistinguishable from each other. Thus it is not clear how to determine to which frequency the dynamics is phase-locked by examining the attractor. On the other hand, phase-slips tend to occur in the region of low intensity when all three phase differences are advancing and of high intensity when all three phase differences are decreasing.

# 5.6 Competition for Phase Synchronization Discussion

We have observed phase synchronization of the intensity dynamics of a chaotic laser driven by two periodic signals at different frequencies. The laser phase alternately locks and slips between synchronization to the two driving frequencies and their average. These behaviors predicted in [51] can be seen and distinguished by plotting one phase difference against the other, even though they are difficult to discern by examining the attractor or a spectrogram.

We believe that competition between two or more frequencies to entrain the dynamics of nonlinear systems is quite likely a common occurrence in natural and engineered systems. Moreover, this itself is a piece of the more general question: How does a system react to a complicated drive? A more complicated drive could be the net result of one or more simpler signals.

Stairwell plots are adequate for observing phase synchronization when drives of two different but close together frequencies perturb the laser. However it is difficult to quantify the degree or durations of the synchronization. Further there is the ambiguous situation where the system appears to entrain to two frequencies simultaneously.

The stairwell plot method for observing phase synchronization has two other shortfalls. First, the frequencies of interest must be known by some other method. Second, in the case where three or more sinusoidal signals drive the laser, it is not known how to observe and distinguish between locking to each frequency. Perhaps methods that rely on constructing intrinsic mode functions, such as the Hilbert-Huang transform [85], would work in this case. Such methods might also provide for a better quantitative analysis of phase synchronization onset and duration.

In the next chapter, we will study the case where the laser is driven by a chaotic signal. In this case, the chaotic signal will be a recording of the laser output intensity.

### Chapter 6

### **Generalized Synchronization**

### 6.1 Introduction

Generalized synchronization can be less apparent than either identical or phase synchronization, as it is characterized only by the existence of some functional relationship between the systems in question [9, 10, 11, 12]. Yet this form of synchronization may be quite common in natural dynamical systems.

Proposed applications of generalized synchronization are typically methods for communication or information encoding, as in Refs. [87, 88]. However, generalized synchronization may be more important in understanding phenomena already occurring in natural systems. In principle, any coupled systems could exhibit generalized synchronization. At least two methods for detecting this kind of synchronization in experimental data have been proposed.

First, if the existence of functional relationship or predictability between the drive and response can be shown to exist, then generalized synchronization is implied. Mutual false nearest neighbors [10] is an algorithm used to determine if close points in the delay-embedded space of a response system correspond to points close to one another in the delay-embedded space of the drive system. Such proximity of points implies a continuous mapping from drive space to response space. An analytical method for approximating such a relationship was introduced in Ref. [86]. Another statistical procedure to detect the mapping has been recently introduced

[11].

Alternative to demonstrating or constructing a functional relationship, when duplicate versions of the response system are accessible, generalized synchronization may be detected directly by observing identical synchronization between the duplicate response systems [12]. This is called the auxiliary system approach.

Generalized synchronization was first reported in Ref. [9] in an experiment with a single-mode ammonia ring laser. In Ref. [9], the system is identical and independently chaotic in both the drive and response configuration. In that experiment, the character of synchronization depends on the strength of the driving signal. For weak driving, generalized synchronization occurs, whereas for strong driving, the chaotic response system identically synchronizes to the drive.

In this chapter, we investigate synchronization in the same Nd:YAG laser used in the previous chapters. This laser feedback signal is recorded and then later replayed to the laser. This procedure simulates coupling two lasers in a driveresponse configuration via the electronic feedback voltage of the driving laser. With only a single laser, the parameters of the "two lasers" are more similar than might otherwise be possible.

Using the feedback signal for coupling allows recording of the feedback signal and replication of that signal at a later time, which is not currently possible with laser light. This signal also has the advantage of being transmitted through electronic cables rather than free space or fiber-optics.

In this experiment, feedback is removed in the response configuration, and the recorded feedback signal is replayed to the laser in exactly the same way that it originally drove the chaotic laser dynamics.

Surprisingly, the laser never identically synchronizes to its previous behavior. Instead, generalized synchronization is observed between the laser drive and response.

Generalized synchronization is verified with the auxiliary system approach of Ref. [12]. The degree of synchronization is measured by comparing the reproducibility of the response, and the quality of synchronization as a function of the bias of the feedback loop is studied. Generalized synchronization is shown to be best when the bias of the drive and the bias of the response are not matched.

The presence of hidden variables within the laser, specifically multiple cavity modes of slightly different frequencies, is proposed as the mechanism responsible for the appearance of generalized, rather than identical, synchronization. Numerical simulations of the Tang-Statz-deMars two-mode laser model [55, 56, 57, 58, 59] show good agreement with the experimental results.

#### 6.2 Drive and Response Setup

Our system is the laser with feedback shown in Fig. 6.1 (a), which is similar to that presented in the preceding chapters.

After recording the feedback signal driving the chaotic dynamics, the same recorded signal can be replayed to the laser in the configuration of Fig. 6.1 (b). In this way, feedback is removed, but the original feedback signal generating the chaotic dynamics of the laser is inserted at the same location within the loop from which it was recorded. The recorded signal is replayed by an arbitrary function generator with 12 bits output resolution and operating at 5 Msamples/s.

The digitization rate of the replayed signal is fifty times faster than the cutoff frequency of the amplifier in the loop. Due to this fast sample rate and the high precision of the recording and playback, the electronic signal driving the laser dynamics during the initial feedback and during replay are essentially identical despite the digital nature of the oscilloscope and function generator.

The design of this experiment differs from that in Ref. [9] in three important ways. First, the Nd:YAG laser used here has multiple cavity modes. Second, in the response configuration (Fig. 6.1 (b)), the laser is not chaotic in the absence of the driving signal. Third, the recorded signal is played at only the same rate as that at which it was recorded.



(b) Send Drive and Record Response



Figure 6.1: a) Experimental setup of a diode-pumped Nd:YAG laser with an intracavity AOM. The line connecting the photodetector to the AOM indicates the feedback loop generating the chaotic dynamics. The bias may be used as a bifurcation parameter to select a variety of laser dynamics. b) With the feedback removed, a signal s(t) from the arbitrary function generator may be played to the laser. Recorded signals are played back to the laser at the location in the loop from which they were recorded.



Figure 6.2: Bifurcation diagram of the probability distribution function (PDF) of the laser intensity signal for 101 values of feedback bias B without control. The grayscale axis represents the log of the probability.

# 6.3 Generalized Synchronization Experimental Results

The most easily adjusted laser parameter is the bias of the feedback loop. Sweeping the bias through a small range causes the laser to produce a variety of waveforms. The dependence of the probability density function (PDF) of the laser intensity on the feedback loop bias is displayed in Fig. 6.2.

From lowest to highest bias values, the laser dynamics range from steady state to low-amplitude chaos to chaotic spiking to periodic spiking. Samples of these waveforms are displayed in Fig. 6.3.

Initially, the laser intensity signal was recorded with the feedback loop in place as in Fig. 6.1 (a). Feedback was removed as shown in Fig. 6.1 (b), and the signal was replayed to the laser by the arbitrary function generator. The dynamics of the



Figure 6.3: Selected time series from the data represented in Fig. 6.2. a) Periodic behavior at a bias voltage of -0.340 V. b) Chaotic oscillations at a bias voltage of -0.350 V. c) Chaotic oscillations at a bias voltage of -0.360 V. d) Chaotic oscillations at a bias voltage of -0.370 V. As the bias voltage becomes more negative, the spikes become first more periodic in time and then in amplitude as well. Experiments performed in this paper use dynamics like those depicted in (b) and (c).



Figure 6.4: A correlation plot of the laser response versus the replayed driving signal. The grayscale is normalized such that the sum of the values of all pixels is 1. The response is not identically synchronized to the drive; the synchronization error Scalculated using Eq. (6.1) is 0.424.

laser with the feedback loop was compared to the laser dynamics without feedback but with driving by the replayed feedback signal.

Instead of displaying identical synchronization as expected, the drive and response signals appeared uncorrelated. Fig. 6.4 shows the correlation between drive and response.

To quantify the synchronization, we calculate the synchronization error S:

$$S = \sqrt{\left\langle \frac{(I_1 - I_2)^2}{(I_1 + I_2)^2} \right\rangle},$$
(6.1)

where the angle brackets denote averaging. Similar to the measure used to determine fringe visibility, the measure S can be thought of as the mean distinguishability of two signals. For nonnegative signals  $I_1$  and  $I_2$ , S ranges between 0 and 1, with 0 indicating indistinguishability (exact correlation). We consider S < 0.10 to indicate some degree of identical synchronization signals  $I_1$  and  $I_2$ . Using the drive as  $I_1$  and the response signal as  $I_2$ , we calculate the synchronization error S = 0.424, indicating that the response is not similar to the driving signal.

When the signal is played back to the laser several times and each response is recorded, it can be seen that there is similarity between the responses. The similarity between the responses can be improved by adjusting the bias to a value different than that at which the signal was recorded.

Figure 6.5 shows an example of a drive signal and two response signals, with the bias tuned to optimize synchronization between responses. Using Eq. (6.1), the synchronization error S between the responses is 0.013, indicating nearly identical synchronization. Note that the response signal is smaller in amplitude than the driving signal.

The recorded driving signal is tens of milliseconds long. Playing the same driving signal on a loop and keeping the bias constant, the laser can remain synchronized in this way to the recorded drive for minutes to hours, orders of magnitude longer than the microsecond time scale of the intensity dynamics. Detecting generalized synchronization by examining the response to the repeated driving signal is analogous to the auxiliary system approach. In this case, we are comparing multiple responses of the same system to the same driving signal, eliminating the need for duplicates of the original system. Figure 6.6 displays the correlation plot between the two responses depicted in Fig. 6.5 (b) and (c).

It may be noticed from Fig. 6.5 that there appears to be phase synchronization between the drive and response. The existence of such phase synchronization is more apparent when the Hilbert phases of the chaotic drive  $\phi_D$  and the phase of the response  $\phi_R$  are calculated as described in [48]. Figure 6.7, displays a plot of the phase difference  $\Delta \phi = \phi_R - \phi_D$  as a function of time. We can see that there are long durations over which  $\Delta \phi$  is nearly constant, indicating phase synchronization.

A reasonable question is, does the quality of generalized synchronization degrade where the drive and response exhibit phase-slips? Fig. 6.8 (a) and (b) compares the phase synchronization between drive and response for the two responses



Figure 6.5: a) A portion of a signal generated by the laser with feedback (Fig. 6.1 a). The length of the entire recorded signal was 26 ms. This signal was continuously replayed to the laser with feedback removed (Fig. 6.1 b). b) and c) corresponding portions of two examples of the laser response, with the bias tuned to optimize the repeatability of the response.



Figure 6.6: A correlation plot of one response versus another. The grayscale is normalized such that the sum of the values of all pixels is 1. In this case, the responses are identically synchronized with synchronization error S = 0.013, indicating that the response exhibits generalized synchronization to the drive.



Figure 6.7: The phase difference  $\Delta \phi_A = \phi_{RA} - \phi_D$  between laser response A and the drive signal as a function of time. Regions where  $\Delta \phi$  is a horizontal line represent phase synchronization between driven and response.

displayed in Fig. 6.5 (b) and (c) in the region of a phase-jump. Fig. 6.8 displays the local synchronization error  $S(t) = (I_A(t) - I_B(t))/(I_A(t) + I_B(t))$  over that time. The local synchronization error does not appear to be affected by the phase-jump. Surprisingly, the phase dynamics of the responses appear to be well-synchronized even across the phase-jump.

When the recorded signal is played back to the laser without changing any laser parameters, generalized synchronization is noticeable but its quality is poor. Figure 6.9 shows the variation of synchronization error as the bias is changed. Generalized synchronization error is minimized for a value of bias near to, but not matching, that at which the driving signal was recorded. Identical synchronization between drive and response is never observed.

The experiment has imitated feedback coupling of two lasers. In this case, there is really only one laser, with the response being recorded within a few minutes after the driving. Thus these "two lasers" are as nearly identical as possible in this experiment. Yet they do not synchronize identically when the feedback voltage of the drive is applied to the response system in lieu of its own feedback. Rather, the response laser synchronizes in a generalized sense to the driving laser and this synchronization is improved by slightly mismatching the drive and response parameters. The direction of the bias mismatch required for improving generalized synchronization is repeatable and matches the model results, indicating that this is not an effect of internal parameters of the laser system drifting gradually in time.

Ideally, the only differences between the driving and response systems are the initial conditions of the laser. The repeatability of the response implies the existence of a stable synchronization manifold. That this synchronization manifold is not identical to the original dynamics suggests that hidden degrees of freedom play a role in determining the shape of the synchronization manifold.

While the laser intensity and the voltage of the feedback loop are known, the population inversion within the laser crystal is not. However, simulation of the laser model with three-equations, one each for the intensity, population inversion, and feedback voltage, as presented in Ref. [83], produces only identical synchronization.



Figure 6.8: a) The phase difference  $\Delta \phi_A$  between laser response A and the drive signal as a function of time, displaying a phase-jump around t = 1.28 ms. b) The phase difference  $\Delta \phi_B$  between laser response B and the drive signal over the same time, displaying great similarity to that of response A. c) The local synchronization error between the responses. There does not appear to be an increase in synchronization error associated with the phase-jump. Instead, the two responses appear to have synchronized phase dynamics even across the jump.



Figure 6.9: Synchronization error S (Eq. (6.1)) as a function of bias detuning in the experiment. Bias detuning refers to the difference between the bias voltage applied while recording the driving signal and the bias voltage applied while replaying the signal to the laser in the receiving configuration (see Fig. 6.1). Synchronization error is lowest for bias detuning of 0.0127 V.

Multiple lasing modes within the cavity have been observed with a scanning Fabry-Perot confocal interferometer, as shown in Fig. 2.3 on page 14. These modes provide additional degrees of freedom. The existence of these modes along with the inability of the single-mode model to produce generalized synchronization motivates the use of a two-mode model.

## 6.4 Generalized Synchronization in the Tang-StatzdeMars Two-Mode Model

We use the Tang-Statz-deMars (TSD) two-mode model presented as Eq. 3.12 in Chapter 3 on page 29 to simulate the dynamics of our laser system. This model is reprinted here for convenience:

$$\frac{dn_0}{dt} = w_0 - n_0 - \gamma_1 \left( n_0 - n_1/2 \right) s_1 - \gamma_2 \left( n_0 - n_2/2 \right) s_2,$$

$$\frac{dn_1}{dt} = n_0 \gamma_1 s_1 - n_1 \left( 1 + \gamma_1 s_1 + \gamma_2 s_2 \right),$$

$$\frac{dn_2}{dt} = n_0 \gamma_2 s_2 - n_2 \left( 1 + \gamma_1 s_1 + \gamma_2 s_2 \right),$$

$$\frac{ds_1}{dt} = K \left[ \gamma_1 \left( n_0 - n_1/2 \right) - 1 - a \sin^2 \left( z \right) \right] s_1,$$

$$\frac{ds_2}{dt} = K \left[ \gamma_2 \left( n_0 - n_2/2 \right) - 1 - a \sin^2 \left( z \right) \right] s_2,$$

$$\frac{dz}{dt} = -\beta \left[ z - B - f \left( s_1 + s_2 \right) \right],$$
(6.2)

 $n_0$  is the space-averaged component of population inversion density with spatial hole burning, normalized by the threshold value.  $n_1$  and  $n_2$  are the first spatial Fourier components of population inversion density with spatial hole burning for the two longitudinal modes.  $s_1$  and  $s_2$  are the normalized amplitudes of the lasing intensities for the two modes.  $w_0 = 1.1$  is the optical pump parameter scaled to the laser threshold.  $\gamma_1 = 1.00$  and  $\gamma_2 = 0.98$  are the gain coefficients for the two modes.  $K = \tau/\tau_p = 36840$ , where  $\tau = 0.23 \ \mu s$  is the upper state lifetime and  $\tau_p$  is the photon lifetime in the laser cavity.

The sixth equation, for the voltage z applied to the AOM driver, is added to

the normal TSD model.  $\beta = 144$  is the cutoff frequency of the filter (corresponding to 100 kHz). The bias of the feedback loop is represented by B, a = 0.015 is the amplitude of the loss modulation, and the amplitude of the photodiode response is f = 0.75. Time for the equations is scaled by  $\tau$ . The simulation does not account for the delay between the photodiode measurement of the laser intensity and the AOM response. Numerical results are calculated with the 4th-order Runge-Kutta-Gill method.

As the photodiode does not distinguish between the laser modes, it only picks up the total intensity  $I = s_1 + s_2$ . When recorded driving is applied to the laser model, the recorded signal I(t) replaces the terms  $s_1 + s_2$  in the equation for z. We have assumed that the loss modulator affects each mode intensity equally.

A bifurcation diagram of the laser model, corresponding to the experimental figure, Fig. 6.2, is displayed in Fig. 6.10. The experimental and simulated bifurcation diagrams are quite similar.

A system of equations simulating one driving laser with feedback and two independent response lasers can be constructed by making three copies of the set of Eqs. (3.12). The first set appears exactly as in Eqs. (3.12). The z equations for the second and third set are modified by replacing the  $s_1 + s_2$  terms with the  $s_1 + s_2$ from the driving set of equations. Thus driving replaces feedback for the receiving lasers, each of which is simulated by equations of the form,

$$\frac{dn_{0R}}{dt} = w_0 - n_{0R} - \gamma_1 \left( n_{0R} - n_{1R}/2 \right) s_{1R} 
-\gamma_2 \left( n_{0R} - n_{2R}/2 \right) s_{2R}, 
\frac{dn_{1R}}{dt} = n_{0R} \gamma_1 s_{1R} - n_{1R} \left( 1 + \gamma_1 s_{1R} + \gamma_2 s_{2R} \right), 
\frac{dn_{2R}}{dt} = n_{0R} \gamma_2 s_{2R} - n_{2R} \left( 1 + \gamma_1 s_{1R} + \gamma_2 s_{2R} \right), 
\frac{ds_{1R}}{dt} = K \left[ \gamma_1 \left( n_{0R} - n_{1R}/2 \right) - 1 - a \sin^2 \left( z \right) \right] s_{1R}, 
\frac{ds_{2R}}{dt} = K \left[ \gamma_2 \left( n_{0R} - n_{2R}/2 \right) - 1 - a \sin^2 \left( z \right) \right] s_{2R}, 
\frac{dz_R}{dt} = -\beta \left[ z_R - B_R - f \left( s_1 + s_2 \right) \right],$$
(6.3)

where  $s_1$  and  $s_2$  are the mode intensities of the driving laser. The bias  $B_R$  for the



Figure 6.10: Bifurcation diagram of the probability distribution function (PDF) of the simulated laser intensity signal for 101 values of feedback bias B without control. The grayscale axis represents the log of the probability. The diagram is quite similar to that for the experiment, Fig. 6.2.

receiver can be controlled independently from the bias B for the drive. All three sets of equations are started from different initial conditions.

A section of typical drive and response dynamics is presented in Fig. 6.11. The response and drive are generally synchronized, as evident by the identical synchronization (after a transient) of the two independent responses.

Correlation plots comparing the response and drive and two independent responses are shown in Fig. 6.12. As in the experiment, the response is not identically synchronized to the drive. However, again, two responses are identically synchronized to one another, indicating generalized synchronization between response and drive.

Examining the intensities of the two laser modes of the drive separately, as in Fig. 6.13, we notice that dynamics of the two modes  $s_1$  and  $s_2$  follow entirely different attractors. Yet, aside from a scale factor, the dynamics of the response laser modes are almost identical to one another. Figure 6.14 displays the dynamics of the individual modes of the response laser. Thus the difference between the modes in the drive is the variable hidden from the response system.

In the case of the data presented in the preceding simulation figures, there was no mismatch between the bias of the driving laser and the responding laser. Fig. 6.15 shows the dependence on bias detuning of both the synchronization error S between the responses and the largest conditional Lyapunov exponent. It is not surprising that S for the simulation can be much smaller than that for the experiment, as noise is not included in the simulation.

The conditional Lyapunov exponent is calculated in the normal manner for calculating the largest Lyapunov exponent, suggested in Refs. [89, 90]. We repeatedly evaluate the variational equations (Eqs. (6.4)) for the receiving laser along the trajectory for a short period of time.

$$\frac{d\delta n_0}{dt} = (-1 - \gamma_1 s_1 - \gamma_2 s_2)\delta n_0$$
$$-(\gamma_1 s_1/2)\delta n_1 - (\gamma_2 s_2/2)\delta n_2$$
$$-\gamma_1 (n_0 - n_1/2)\delta s_1 - \gamma_2 (n_0 - n_2/2)\delta s_2$$



Figure 6.11: a) A portion of the simulated laser intensity  $s_1 + s_2$  for the driving laser (Eqs. (3.12) for B = 1.44. b) and c) Independent simulations of the laser response intensity, each started from different initial conditions.



Figure 6.12: a) Correlation plot of the simulated laser response versus the replayed driving signal. The grayscale is normalized such that the sum of the values of all pixels is 1. The response is not identically synchronized to the drive; the synchronization error S calculated using Eq. (6.1) is 0.459. b) Correlation plot of two independent simulated laser responses to the drive. The synchronization error S between the responses is 0.000, indicating that the drive and response are generally synchronized. In this case the bias B for the drive is matched to that of the response,  $B_R$ .



Figure 6.13: a) Plot of the total intensity  $s_1 + s_2$  of the simulated laser dynamics of the drive laser, Eqs. (3.12). b) Plot of the intensity  $s_1$  of mode 1 only. c) Plot of the intensity  $s_2$  of mode 2 only.  $B_R = B = 1.44$ . The two modes display very different behavior.



Figure 6.14: a) Plot of the total intensity  $s_{1R} + s_{2R}$  of the simulated laser dynamics of the response laser, Eqs. (6.3). b) Plot of the intensity  $s_{1R}$  of mode 1 only. c) Plot of the intensity  $s_{2R}$  of mode 2 only, enlarged by a factor of 4 with respect to (b). Aside from a scale factor, the dynamics of the two modes is essentially identical.



Figure 6.15: The dotted line represents the synchronization error S (Eq. (6.1)) as a function of the bias detuning  $B - B_R$  between the drive (Eqs. (3.12)) and the response (Eqs. (6.3)) in the numerical experiment. The solid line represents the conditional Lyapunov exponent (see text) between two responses. Synchronization becomes stable where the conditional Lyapunov exponent is below zero (the dotted line). Stability occurs around bias matching, but the conditional Lyapunov exponent is most negative for bias mismatch around 10 mV. The horizontal axis has been scaled to match voltage in the experiment. Near a bias mismatch value of 13 mV, there are two data points where the conditional Lyapunov exponent is positive but the synchronization error is still near zero. The algorithm used to calculate the conditional Lyapunov exponent is very sensitive to algorithm parameter choices where synchronization is near neutral stability.
$$\frac{d\delta n_1}{dt} = \gamma_1 s_1 - (1 + \gamma_1 s_1 + \gamma_2 s_2) \delta n_1 
+ \gamma_1 (n_0 - n_1) \delta s_1 - \gamma_2 n_1 \delta s_2 
\frac{d\delta n_2}{dt} = \gamma_2 s_2 - (1 + \gamma_1 s_1 + \gamma_2 s_2) \delta n_2 
- \gamma_1 n_2 \delta s_1 + \gamma_2 (n_0 - n_2) \delta s_2 
\frac{d\delta s_1}{dt} = K \gamma_1 s_1 \delta n_0 - (K \gamma_1 s_1/2) \delta n_1 
+ [K \{ \gamma_1 (n_0 - n_1/2) - 1 \} + Ka \sin^2(z)] \delta s_1 
+ 2 s_1 Ka \sin(z) \cos(z) \delta z 
\frac{d\delta s_2}{dt} = K \gamma_2 s_2 \delta n_0 - (K \gamma_2 s_2/2) \delta n_2 
+ [K \{ \gamma_2 (n_0 - n_2/2) - 1 \} + Ka \sin^2(z)] \delta s_2 
+ 2 s_2 Ka \sin(z) \cos(z) \delta z 
\frac{d\delta z}{dt} = -B \delta z$$
(6.4)

At the end of each evaluation, the norm  $N = \sqrt{\delta n_0 + \delta n_1 \delta n_2 + \delta s_1 + \delta s_2 + \delta z}$  of the variation is recorded. Next, the variations are rescaled by N, and integration of the trajectory resumes. The conditional Lyapunov exponent is taken to be the mean of the values of N recorded at the end of each evaluation.

Negative values for the conditional Lyapunov exponent indicated that the synchronization manifold is stable. Positive values indicate an unstable synchronization manifold, and zero indicated neutral stability.

As the bias is detuned between the simulated laser in drive and response configurations, the synchronization error shown in Fig. 6.15 drops almost to zero near the matching condition and remains low for bias detuning of up to a few percent. As expected, the conditional Lyapunov exponent becomes negative (indicating a stable synchronization manifold) in the same region. The experiment behaves in a similar manner, as depicted in Fig. 6.9. There, the synchronization error dips around zero bias detuning, and remains low for small positive bias detuning.

# 6.5 Discussion

Identical synchronization is not necessarily the ultimate case of generalized synchronization, as suggested in Ref. [9]. In this experiment, identical synchronization between drive and response is never observed. However, generalized synchronization is observed throughout a small range of parameter mismatching, and synchronization is optimized for mismatched laser parameters.

The presence of more than one cavity mode is believed to be responsible for the absence of identical synchronization between the laser in response mode and its dynamics in drive mode. A single-mode laser model shows only identical synchronization while a two-mode laser model displays generalized synchronization without identical synchronization.

The multiple cavity modes of the laser can be observed independently. Doing so should allow for refinement of the model and verification that the mode structure is responsible for the generalized synchronization.

### Chapter 7

# Conclusion

The laser is already considered a powerful tool as a stable source of bright, coherent light. However, the dynamical behavior of the laser also has many applications.

In this dissertation, we presented the construction of Nd:YAG laser with feedback. Single-mode and two-mode laser models were described and analyzed. The chaotic intensity dynamics of the laser were explored. Phenomena such as phase entrainment and generalized synchronization arose from the perturbation of the laser with signals varying in complexity from periodic to chaotic.

#### 7.1 Design and Modeling

In Chapter 2, the design of the Nd:YAG laser and feedback loop was described. The physical characteristics of the laser and its components were detailed in Section 2.1. The laser has several control parameters, including the pump power and the bias of the feedback loop.

Several models for the laser were presented in Chapter 3. The Class-C Maxwell-Block single-mode laser equations in terms of electric field, population inversion, and atomic polarization were discussed in Section 3.2. There, the optical lasing frequency was solved for and the Class-B equations are derived in terms of electric field and population inversion were derived by adiabatic elimination. The model was further transformed into terms of intensity and population inversion. A single-mode model for the laser with feedback was presented in Section 3.3. The laser model was renormalized into a form useful for numerical simulation and analytical manipulation. In section 3.4, the steady-state solution of the laser and the relaxation oscillation frequency of the laser were solved for. A means for characterizing the cavity loss of the laser by observing the dependence of the relaxation oscillation frequency with respect to the pump power was calculated.

In Section 3.5, the two-mode Tang-Statz-deMars laser model was presented. This model accounts for the distinct spatial distribution of the modes through the gain medium.

#### 7.2 Chaotic Function Generation

In Chapter 4, we have explored some of the dynamics that an Nd:YAG laser with feedback is capable of producing. Even though exploration has not been exhaustive, we have shown that the laser can generate a wide range of intensity waveforms.

The design of the chaotic function generating laser was presented in Section 4.3. This design included an additional loop of the feedback network designed to suppress the higher frequency components of the feedback that give rise to the highest-amplitude dynamics.

We consider the laser to be, among other things, a tunable generator of chaotic waveforms. The diversity of these waveforms has been characterized by statistical properties such as probability distribution functions, power spectra, and Lyapunov exponents. Such data can be collected in tables or in bifurcation diagrams, as shown in Section 4.4. The bifurcation diagrams can later be used in the standard way as a reference to characterize the laser. These data can also be used as a map, i.e. to determine the region of parameter space in which the laser is likely to emit waveforms satisfying some desired criteria.

The single-mode model of the laser with feedback and auxiliary control loop was presented in Section 4.5. Within a small range of parameters, the model showed chaos similar to that displayed by the laser over a much larger range. Potential uses for such waveform generators include analog simulators of other, less accessible, physical systems and sources of chaotic drive signals.

## 7.3 Phase Synchronization

In Chapter 5, one- and two-frequency driving is added to the feedback. The entrainment of the phase of the laser dynamics is explored.

In Section 5.2, modifications to the laser setup that allow for external driving were presented. The Hilbert phase for the chaotic signal was constructed as shown in Section 5.3. When perturbed by an external periodic signal of sufficient amplitude, the laser dynamics phase synchronize to the perturbation as shown in Section 5.4.

The laser is exposed to driving by two periodic signals with slightly differing periods, as reported in Section 5.5. In this case, competition between those frequencies to entrain the dynamics becomes evident. The system also displays phase-entrainment to the average of the driving frequencies.

Competition for synchronization between the two driving frequencies and their average can be viewed by plotting one phase difference against another, in what Breban and Ott call a staircase plot [51]. However, this representation requires a priori knowledge of the frequencies to examine, and only two frequencies and their average can be distinguished.

#### 7.4 Generalized Synchronization

In Chapter 6, the laser feedback signal responsible for producing the chaotic dynamics was recorded and replayed to the laser without feedback. The description of the setup for this experiment was given in Section 6.2.

Although the laser received exactly the same signal as that which created the recorded chaos, the laser did not identically synchronize to its original dynamics. Instead, the laser produced chaotic dynamics on a new attractor. These dynamics were, however, generally synchronized to the original laser dynamics, as demonstrated by the repeatability of the response to repetitions of the same driving signal. These results have been presented in Section 6.3.

In addition to generalized synchronization, phase synchronization between drive and response with occasional slips was also displayed. The generalized synchronization error did not appear to increase in the neighborhood of the phase slips. Rather, phase slips between drive and response appeared to be well synchronized. The phase slips are most likely an inherent component of the generalized synchronization rather than being induced by noise.

The absence of identical synchronization together with the presence of generalized synchronization can be explained by the existence of multiple cavity modes within the laser. These modes provide a kind of "hidden variable" in the feedback because the photodiode providing the feedback signal does not distinguish between cavity modes. The two-mode Tang-Statz-deMars laser model was used in Section 6.4 to simulate the laser dynamics.

Numerical simulations of this two-mode model display agreement with the experiment far superior to that of the single-mode model presented in Section 4.5. In Section 6.4, the model bifurcation diagram agrees with the experimental diagram across a wide range of feedback loop bias values. The time series show nearly identical dynamics, and generalized synchronization is observed and compares well with that of the experiment. In the model, the laser modes of the drive follow distinct attractors whereas the laser modes of the response follow similar attractors.

#### 7.5 Further Research

Many of the topics that might be addressed in future research revolve around the nature, accuracy, and applications of synchronization.

Empirical mode decomposition techniques, such as the Hilbert-Huang transform described in Ref. [85], allow for the construction of basis functions with timevarying frequencies. The Hilbert-Huang transformation has already been shown to be useful for analyzing data in highly complicated systems such as weather-related phenomena [91] and electroencephalogram data [92]. Construction of the Hilbert phase used in Chapter 5 relies on Fourier decomposition with basis functions that are stationary in time. The frequency of the empirical modes Huang uses vary in time. Thus it should be possible to observe competition for phase synchronization between multiple driving frequencies with greater time resolution and without any knowledge ahead of time of the frequencies involved by using an empirical mode method.

Does the phase or generalized synchronization error show dependence on location in the attractor? Although phase-slips between drive and response in the generalized synchronization case do not appear to play a role in loss of generalized synchronization, it is possible that the local synchronization error is dependent upon some other factor, such as location within the attractor. Little is currently known about the stability of phase or generalized synchronization manifolds. Further, one might imagine that the ability to influence the stability of complicated synchronization manifolds has applications to problems including the treatment or prevention of sleep disorders, epilepsy, heart attacks, epidemics, and extinctions.

One might consider implementing message encoding and transmission by generalized synchronization as suggested by Kocarev [87] or Terry [88]. In Chapter 6, generalized synchronization has been shown to endure in the face of small perturbations of a control parameter; perhaps this durability alone is enough to allow communication.

Can the functional relationship of the generalized synchronization be determined by methods such as that proposed by Brown [86]? If this relationship can be solved for or approximated, then the study of the factors influencing the nature of the generalized synchronization function is accessible.

These and many other questions do not need to be addressed with this or any particular laser system. It is the commonality of their occurrence that makes chaos and synchronization interesting and useful.

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