Homework # 1 Solutions

6.1.10(a)

\[
\sin(x + iy) = \frac{1}{2i} \left[ e^{ix} e^{iy} - e^{-ix} e^{-iy} \right]
\]

\[
= \frac{1}{2i} \left[ e^{-y} (\cos x + i \sin x) - e^{y} (\cos x - i \sin x) \right]
\]

\[
= \frac{1}{2} \left[ \sin x e^{y} + \sin x e^{-y} + i(\cos x e^{y} - \cos x e^{-y}) \right]
\]

\[
= \sin x \cos hy + i \cos x \sin hy
\]

Similar for \(\cos(x + iy)\)

6.1.15

a) Find the zeros of \(\sin z\). From above

\[
\sin z = \sin x \cos hy + i \cos x \sin hy
\]

\[
\Rightarrow \text{must have both real and imaginary part zero.}
\]

Since \(\cos hy > 0\) everywhere,

\[
\sin x = 0 \Rightarrow x = \pi n \text{ integer}
\]

Since \(\cos x = \cos \pi n \neq 0\), must have

\[
\sin hy = 0
\]

\[
\Rightarrow y = 0
\]
a) Find zeros of \( \cosh z \).

\[
\cosh z = \cosh x \cos y + i \sinh x \sin y
\]

Since \( \cosh x > 0 \) \( \Rightarrow \) \( \cos y = 0 \) \( \Rightarrow \) \( y = (n+\frac{1}{2})\pi \)

Since \( \sin (n+\frac{1}{2})\pi \neq 0 \), must have \( \sinh x = 0 \)

\( \Rightarrow x = 0 \)

6.2.1 a)

For \( w = u + iv \) with \( w \) analytic, show that \( \nabla^2 u = \nabla^2 v = 0 \)

CR conditions: \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \)

\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\partial x} \left( \frac{\partial v}{\partial y} \right) + \frac{1}{\partial y} \left( -\frac{\partial v}{\partial x} \right)
\]

\( = 0 \)

\( \nabla^2 u \) similar

6.2.2 Is \( f(z) = \text{Re}(z) = x \) analytic?

\( \Rightarrow u = 0 \)

CR requires \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \) with \( u = x \)

but \( \frac{\partial u}{\partial x} \neq 0 \) so not analytic anywhere.
For \( w = u + iv \) with \( w \) analytic,
show that \( u, v \) can not have max or min in region where \( w \) analytic.

**Proof 1**

Suppose \( u \) has max at \( z_0 \). Taylor series around this point

\[
U(x, y) = U(x_0, y_0) + \frac{\partial U}{\partial x_0} (x-x_0)
+ \frac{\partial^2 U}{\partial y_0^2} (y-y_0) + \frac{1}{2} \frac{\partial^2 U}{\partial x_0^2} (x-x_0)^2
+ \frac{1}{2} \frac{\partial^2 U}{\partial y_0^2} (y-y_0)^2 + \frac{\partial^2 U}{\partial x_0 \partial y_0} (x-x_0)(y-y_0)
\]

Along \( x \) direction:

\[
U(x, y_0) = U(x_0, y_0) + \frac{1}{2} \frac{\partial^2 U}{\partial x_0^2} (x-x_0)^2
\]

Along \( y \) direction:

\[
U(x_0, y) = U(x_0, y_0) + \frac{1}{2} \frac{\partial^2 U}{\partial y_0^2} (y-y_0)^2
\]

If \( u \) has max at \( z_0 \) then \( \frac{\partial^2 U}{\partial x_0^2} < 0 \)
but since \( \nabla^2 U = 0 \) \( \Rightarrow \frac{\partial^2 U}{\partial y_0^2} > 0 \) so along \( y \) direction \( u \) increases

\( \Rightarrow \) no maximum

\( \Rightarrow \) only a saddle point.

**Proof 2**

Use divergence theorem for real functions

\[
\int_{\partial A} \mathbf{n} \cdot \nabla u = \int_{A} \nabla^2 u = 0
\]

A is area within \( \mathcal{Q} \) \( \Rightarrow \) \( \mathbf{n} \cdot \nabla u \) has pos. and neg. values
6.2.8 Find CR conditions in polar coordinates.

\[ f = R(r, \theta) e^{i \theta(r, \theta)} \]

\[ z = re^{i \theta} \]

\[ df = dr e^{i \theta} + R e^{i \theta} id\theta \]

\[ dz = dr e^{i \theta} + r e^{i \theta} d\theta \]

\[ \frac{df}{dz} = \frac{(dr + i r d\theta) e^{i \theta}}{(dr + r i d\theta) e^{i \theta}} \]

In \( r \) direction \( \Rightarrow d\theta = 0 \)

\[ \frac{df}{dz} = \left( \frac{dr}{dr} + i r \frac{d\theta}{dr} \right) \]

In \( \theta \) direction \( \Rightarrow dr = 0 \)

\[ \frac{df}{dz} = \left( -i \frac{1}{r} \frac{d\theta}{d\theta} + r \frac{d\theta}{d\theta} \right) \]

\( \Rightarrow \) must be same in both directions

\[ \frac{dR}{dr} = \frac{R}{r} \frac{d\theta}{d\theta} \]

\( \frac{1}{r} \frac{dR}{d\theta} = -R \frac{d\theta}{d\theta} \)
6.7.1 a) Find how circles in $z$ plane transform.

\[ w = z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta} \]

\[ = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta \]

\[ u = (r + \frac{1}{r})\cos\theta \quad v = (r - \frac{1}{r})\sin\theta \]

\[ \Rightarrow \text{eliminate } \theta \Rightarrow \sin^2\theta + \cos^2\theta = 1 \]

\[ \frac{v^2}{(r - \frac{1}{r})^2} + \frac{u^2}{(r + \frac{1}{r})^2} = 1 \]

In $w$ plane have an ellipse with major and minor axis given by

\[ |r - \frac{1}{r}|, \quad r + \frac{1}{r} \]

As $r \to 1$, becomes a line with $u = 0$ and $|w| \leq 2$.

6.7.3 a)

\[ w(z) = \sin z = \sin x \cosh y + i \cos x \sinh y \]

\[ u = \sin x \cosh y \quad v = \cos x \sinh y \]
For \( y = c_2 \) eliminate \( x \),

\[
\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1
\]

\( \Rightarrow \) ellipses with major, minor axes

\[
cosh c_2 \quad | \quad \sinh c_2
\]

For \( x = c_1 \) eliminate \( y \),

\[
\frac{u^2}{\sin^2 c_1} - \frac{v^2}{\cos^2 c_1} = 1
\]

\( \Rightarrow \) hyperbolas which intersect the \( u \) axis at \( u = \pm \sinh x \)

Note that the contours of \( x \) and \( y \) are \( \perp \) to each other as they cross.

2. Define a cut in \( z \) plane to make \( z^{1/3} \) single valued.

\[
\text{Arg}(-i)^{\frac{1}{3}} = \text{Arg}(e^{-i\frac{\pi}{2}})^{\frac{1}{3}}
\]

\[
= \text{Arg}(e^{-i\frac{\pi}{6}}) = -\frac{\pi}{6}
\]

\( \Rightarrow \) cut not unique.
Define cuts to make
\[ f = \ln (z^2 - 1) \]
single valued. Choose cut with symmetry
\[ \Im \theta = 0 \]
\[ \Rightarrow \text{branch points at } z = \pm 1 \]
\[ \Rightarrow \text{not unique. Can choose any cuts that end at } z = \pm 1 \text{ and extend to } \infty \text{ in any direction.} \]

Above cut with \( x > 1 \) (y \approx 0), define
\[ \text{Arg}(z - 1) = 0 \]
\[ \text{Arg}(z + 1) = 0 \]

At \( @ \), \( \text{Arg}(z - 1) = 0 \), \( \text{Arg}(z + 1) = 0 \)
\[ \Rightarrow \boxed{\Im \theta = 0} \]

At \( @ \), \( \text{Arg}(z - 1) = \pi \), \( \text{Arg}(z + 1) = 0 \)
\[ \Im \theta = \frac{\ln(e^{2\pi i})}{2\pi i} = \Im(\ln e^{2\pi i}) \]
\[ = \boxed{2\pi i} \]

At \( c \), \( \text{Arg}(z - 1) = \pi \), \( \text{Arg}(z + 1) = \pi \)
\[ \Im \theta = \Im(\ln(e^{\pi i})) + \Im(\ln(e^\pi)) \]
\[ = \boxed{2\pi i} \]

At \( a \), \( \text{Arg}(z - 1) = \pi \), \( \text{Arg}(z + 1) = -\pi \), \( \boxed{\Im \theta = 0} \)