Statistical Properties
of Wave Chaotic Scattering Matrices*

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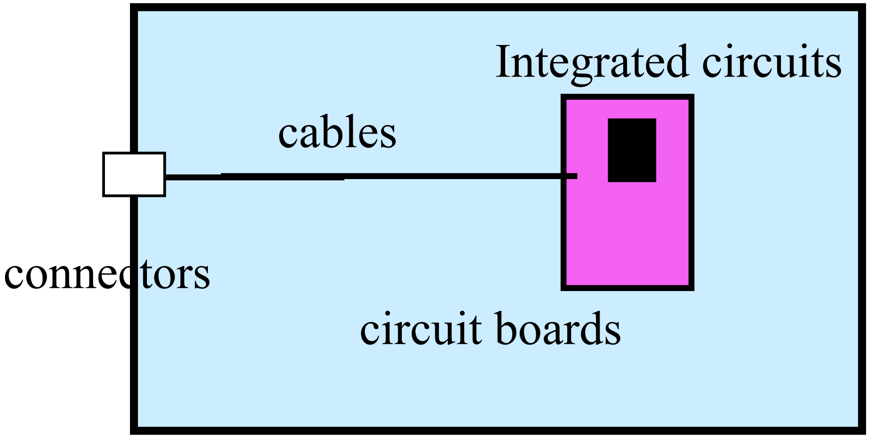
AFOSR-MURI Program Review

Electromagnetic Coupling in Computer Circuits

Schematic



• Coupling of external radiation to computer circuits is a complex processes:

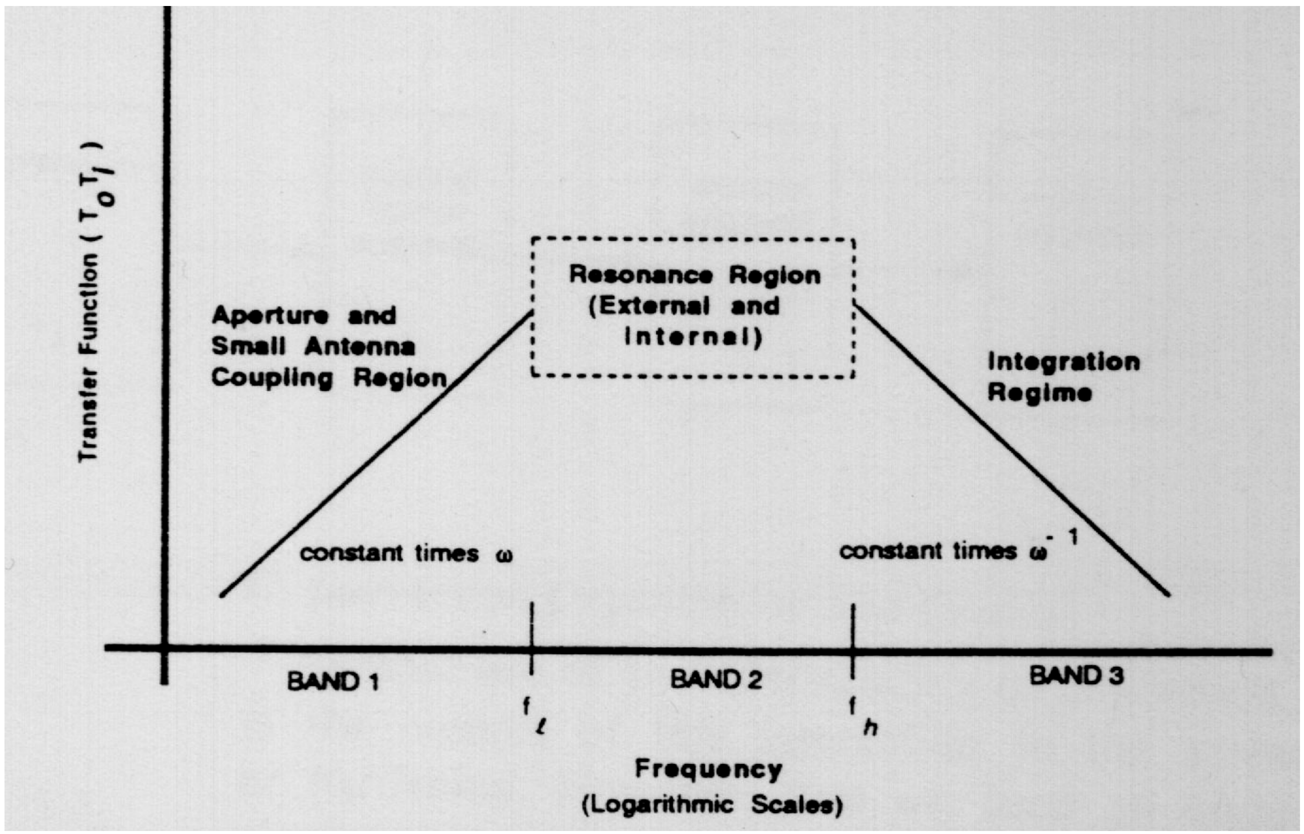
apertures
resonant cavities
transmission lines
circuit elements

• Intermediate frequency range involves many interacting resonances

• What can be said about coupling without solving in detail the complicated EM problem ?

• Wave Chaos

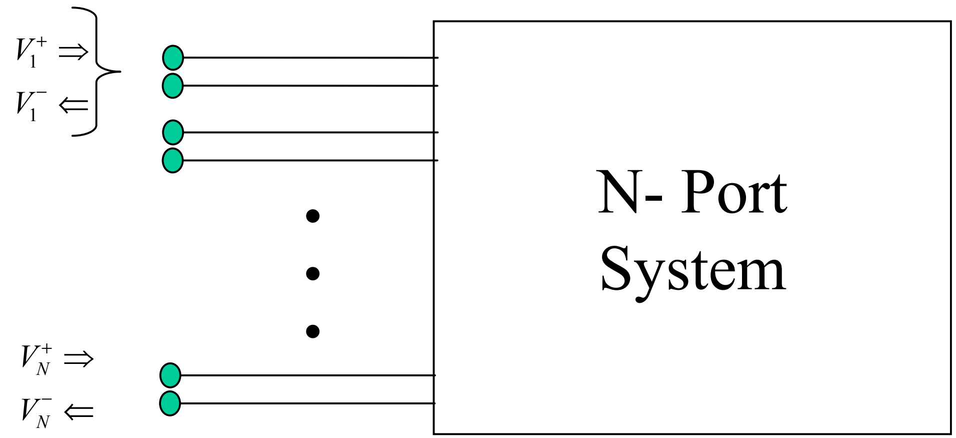
Transfer Function



C. E. Baum, Proc. IEEE <u>80</u>, 789-817 (1992).

S-Matrix

N ports with incoming and outgoing waves



S matrix

$$\begin{pmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_{N1}^- \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_{N1}^+ \end{pmatrix}$$

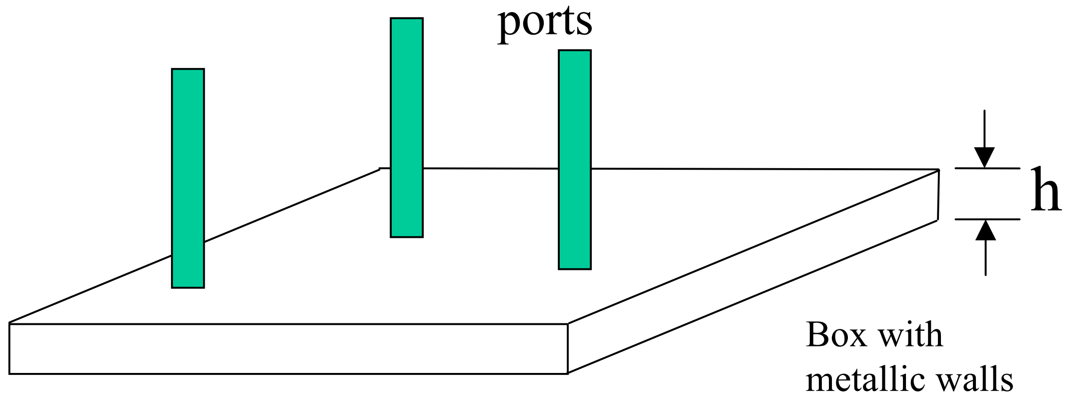
outgoing   incoming

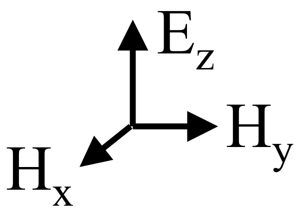
$\boldsymbol{S}(\omega)$

Complicated function of frequency

Details depend sensitively on unknown parameters

Two Dimensional Resonators



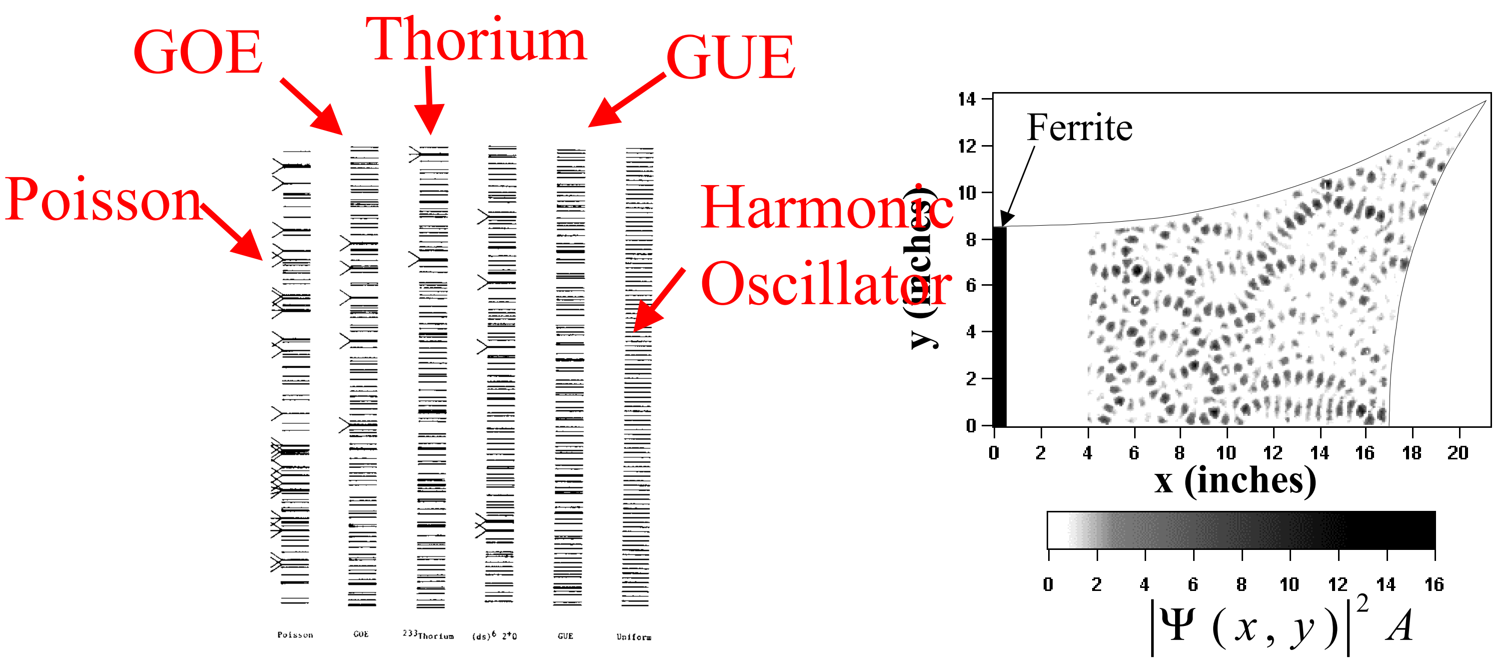


Only transverse magnetic (TM) propagate for $f < c/2h$

Anlage Experiments
Power plane of microcircuit

Wave Chaotic Systems
Statistical Properties

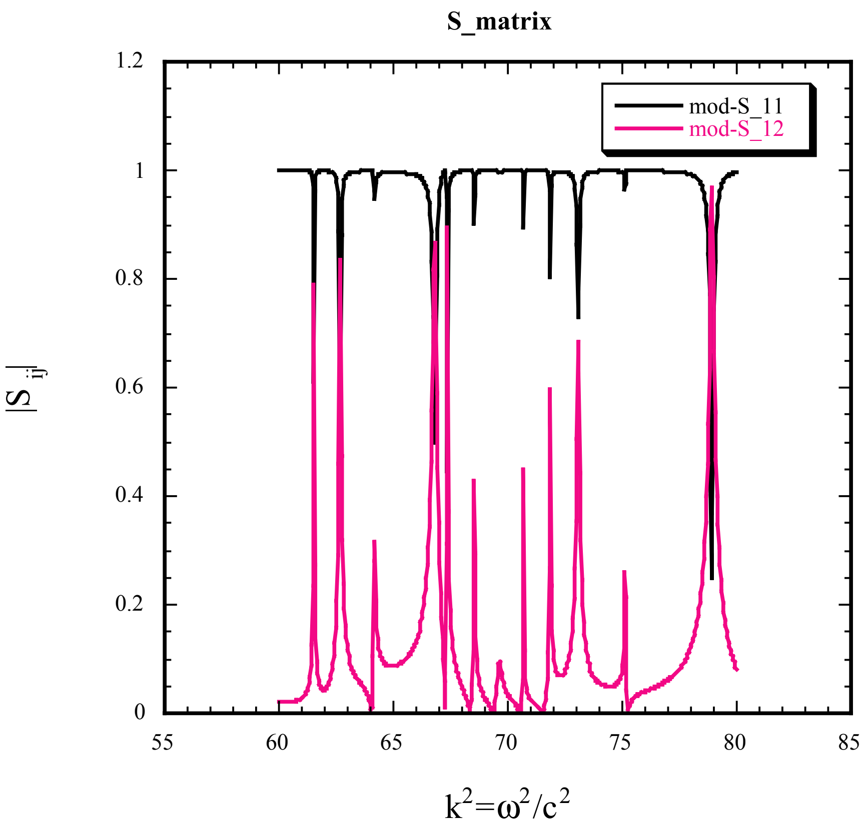
“chaotic” features are not clearly evident in the eigenvalue spectrum and eigenfunctions of wave chaotic systems



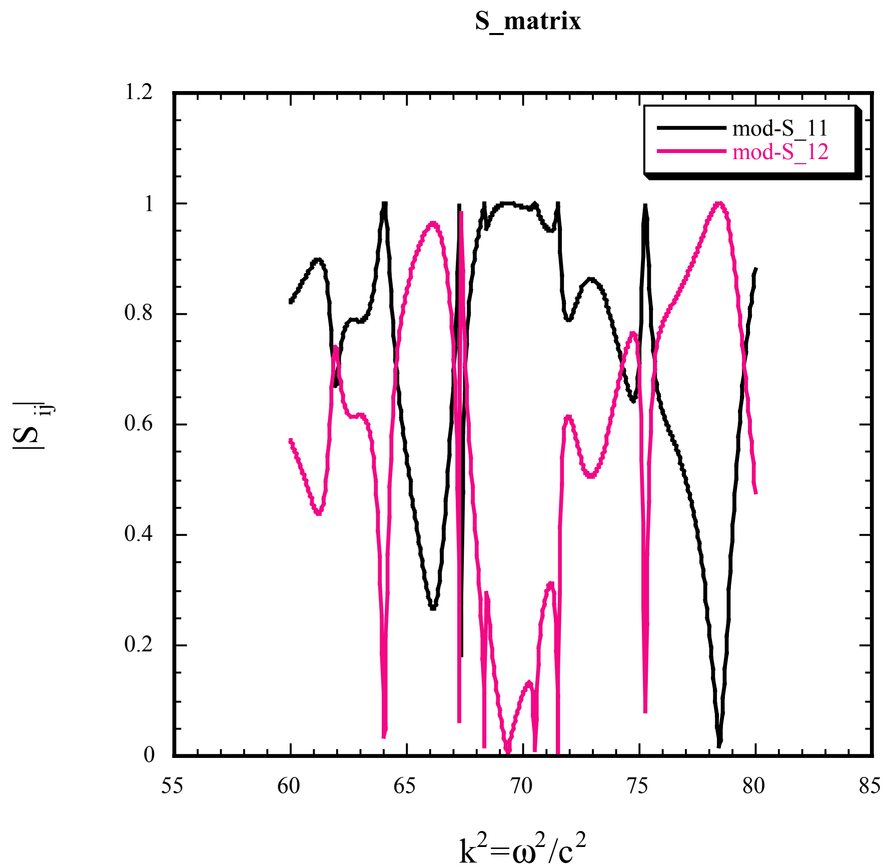
We need to perform statistical analysis of a large number of eigenvalues and eigenfunctions to see the effects of chaos on quantum systems

S- Matrix for Chaotic System

Weak coupling



Strong coupling



Multiple Ports, Inductive Coupling

- Cavity fields driven by currents at ports, (assume $e^{-i\omega t}$ dependence) :

$$\nabla_\perp^2 V_T + k^2 V_T = ikh\eta_0 \sum_{ports} u_i I_{Ci}$$

$$\eta_0 = \sqrt{\mu_0 / \varepsilon_0}$$

$$k = \omega / c$$

Profile of excitation current

- Voltage at j[th] port: $$V_{Cj} = \int dxdy u_j V_T$$

- Impedance matrix $Z_{ij}(k)$: $$Z_{ij} = ikh\eta_0 \int dxdy \quad u_i \left(\nabla_\perp^2 + k^2\right)^{-1} u_j$$

- Scattering matrix: $$\boldsymbol{S}(k) = \left(\boldsymbol{Z} + Z_0 \boldsymbol{I}\right)^{-1} \left(\boldsymbol{Z} - Z_0 \boldsymbol{I}\right)$$

Five Different Methods of Solution

Problem, find: $$\int dxdy \quad u_i \left(\nabla_\perp^2 + k^2\right)^{-1} u_j$$

1. Computational EM - HFSS

2. Experiment - Anlage

3. Random Matrix Theory - replace wave equation with a matrix with random elements

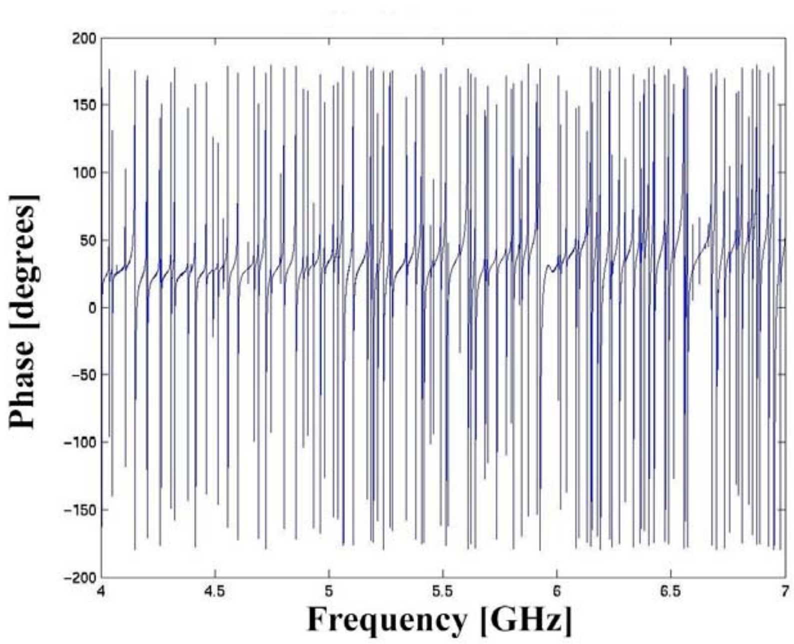
4. Expand in Chaotic Eigenfunctions

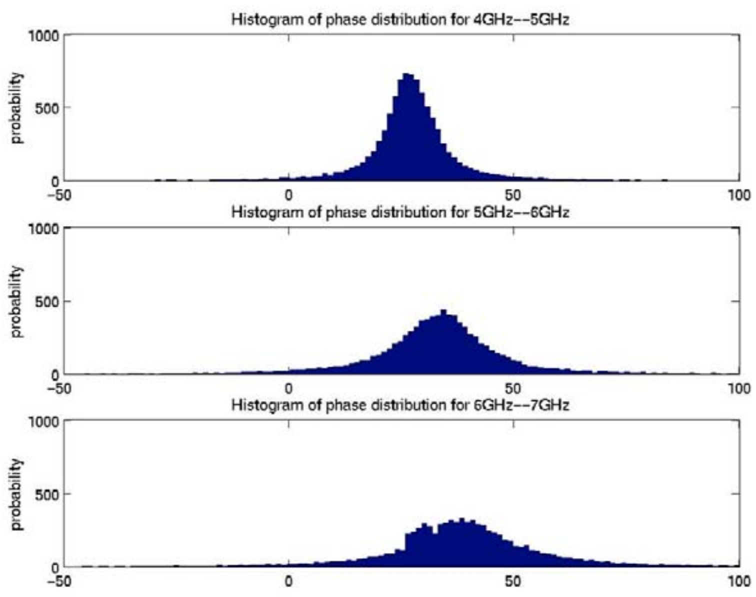
5. Semi-classical - integrate wave packet equations on classical orbits

Phase of S_{11} for Bow Tie from HFSS

Angle vs Frequency

Histogram of Angle





Random Matrix Theory

- Physical problem replaced by random matrices: ***H*** and ***w***

$$Z_{ij} = ikh\eta_0 \int dxdy \quad u_i \left(\nabla_\perp^2 + k^2\right)^{-1} u_j \quad \longrightarrow \quad \gamma \underline{w}^{\ddagger} \left(k^2 \underline{\underline{I}} - \underline{\underline{H}}\right)^{-1} \underline{w}$$

- Elements of ***H*** and ***w*** are Guassian random variables

- Theoretical predictions for many properties of Z and S:
  - $P(Z_{ij})$
  - distribution of poles of S
  - distribution or delay times

Expand in Chaotic Eigenfunctions

$$Z_{ij} = ikh\eta_0 \int dxdy \quad u_i \left(\nabla_\perp^2 + k^2\right)^{-1} u_j = \sum_{\text{modes}-n} \frac{ikh\eta_0 w_{in} w_{jn}}{k^2 - k_n^2}$$

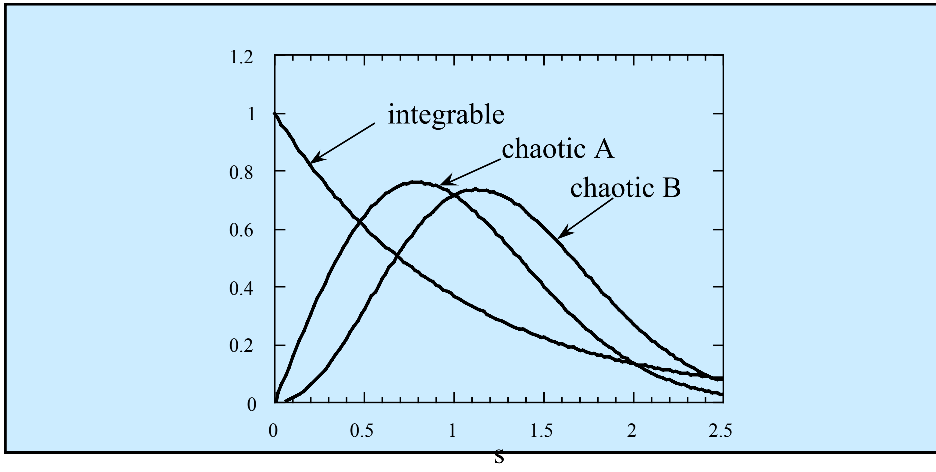
Where:

1. Coupling weights w_{in} are Gaussian random variables

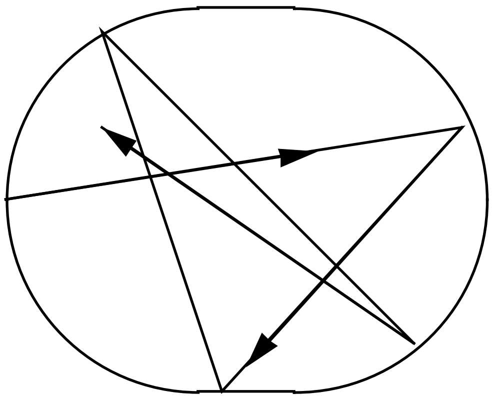
2. Eigenvalues k_n^2 are distributed according to appropriate statictics

Normalized Spacing $\quad s_n = (k_{n+1}^2 - k_n^2) / \langle \Delta k^2 \rangle$

A: time reversible
B: time irreversible
(e.g., due to ferrite)



Chaotic Eigenfunctions



Rays ergodically fill phase space.

Eigenfunctions appear to be a superposition of plane wave with random amplitudes and phases.

$$\psi_\alpha = \lim_{N\to\infty} \mathrm{Re}\left\{\sum_{j=1}^{N} a_j \exp\left[i\left(k_j \cdot x + \alpha_j\right)\right]\right\}$$ reversible

$$\psi_\alpha = \lim_{N\to\infty} \sum_{j=1}^{N} a_j \exp\left[i\left(k_j \cdot x + \alpha_j\right)\right]$$ irreversible

$|\mathbf{k}_j| = k_\alpha$, directions of $\mathbf{k}_j$ randomly distributed

a_j random amplitude, $\langle a_j^2 \rangle = 1$

α_j random phase

ψ_α is a Gaussian random variable

$$P(\psi) \approx \exp\left[-|\psi|^2 / 2\langle|\psi|^2\rangle\right]$$

Properties of Z_{ij}

•Mean and fluctuating parts: $$Z_{ij} = \langle Z_{ij} \rangle + \delta Z_{ij}$$

Mean part: $$\langle Z_{ij} \rangle = 0, \quad i \neq j$$

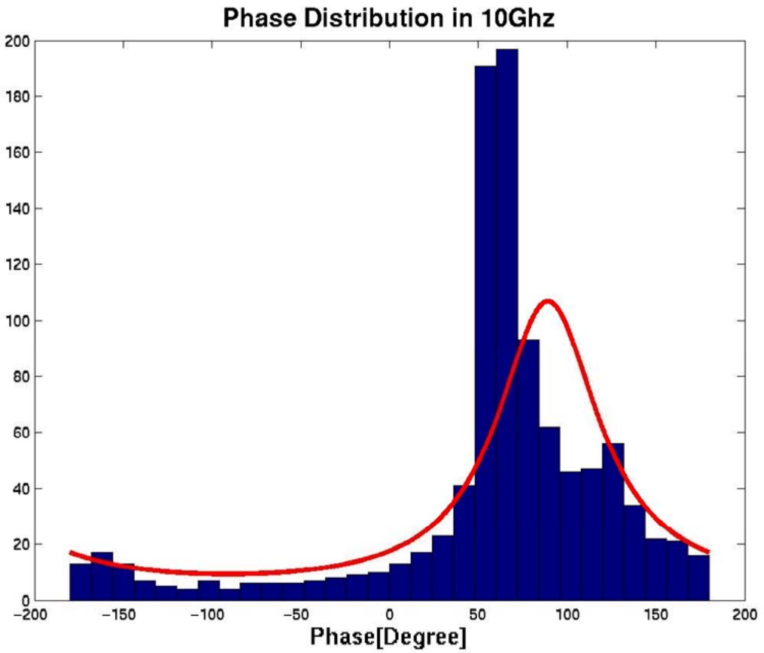
$$\langle Z_{ii} \rangle = \gamma\, P \int_0^{k_{\max}^2} \frac{dk'^2}{k^2 - k'^2} D(k'^2) \langle |w|^2 \rangle \cong \gamma \ln\left(\frac{k_{\max}^2 - k^2}{k^2} \right)$$

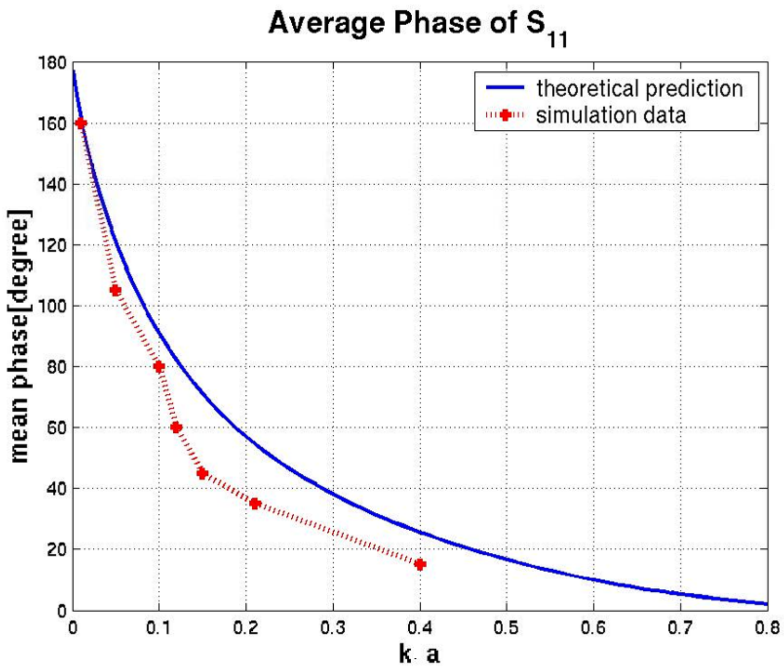
Principle part

Density of states

Coupling strength: $$\gamma = kh \frac{\eta_0}{Z_0}$$

Comparison with HFSS-Single Channel



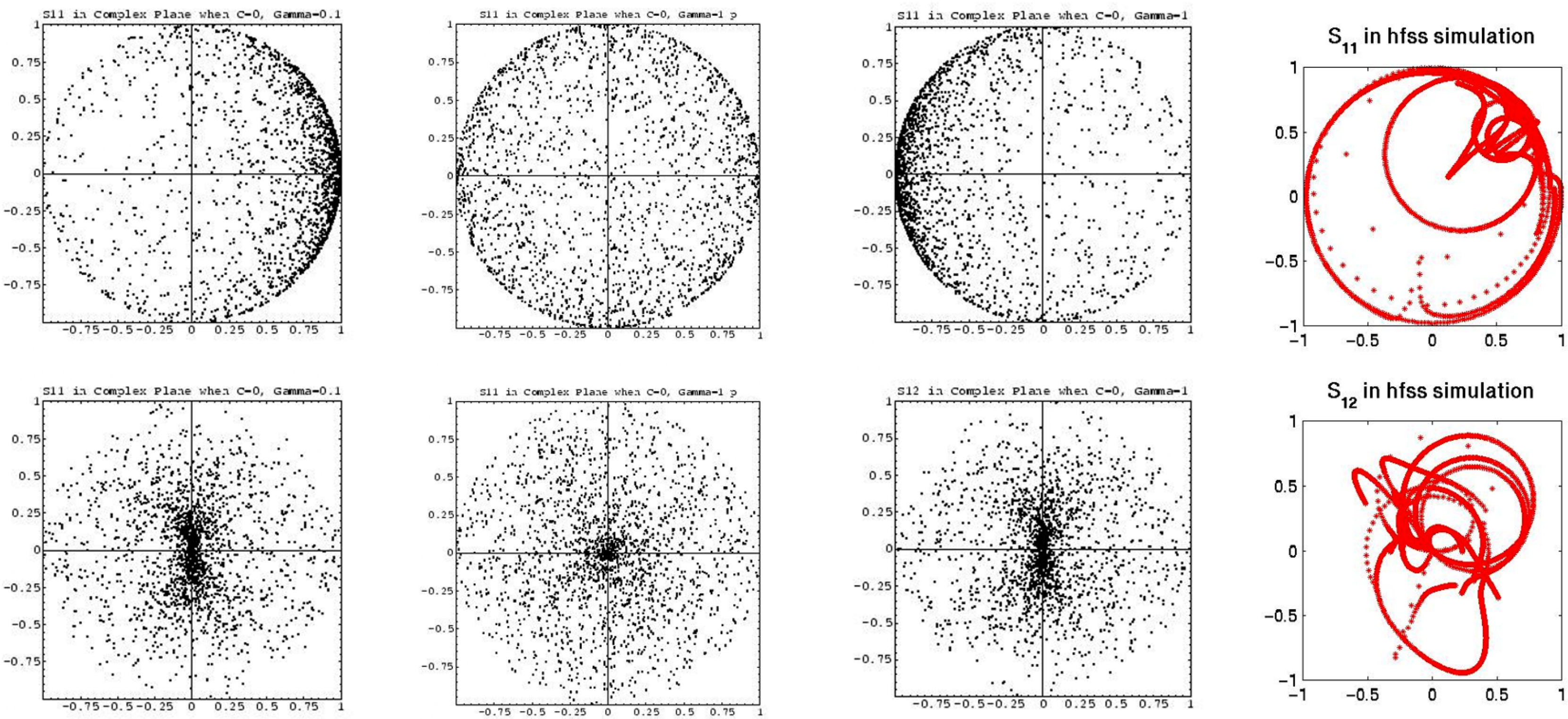


a=radius of center conductor

Sample S-Matrices- 2 Channels

S_{11}

S_{12}



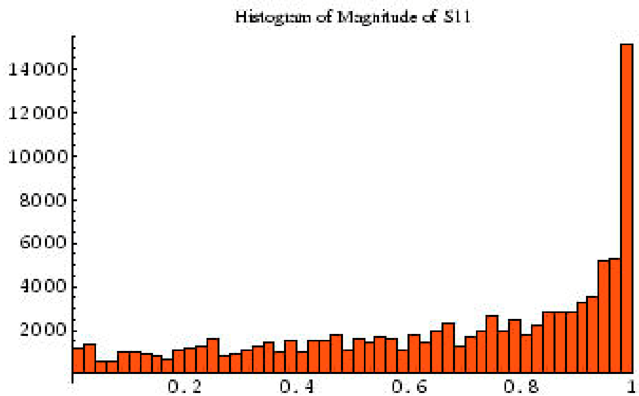
Weak Coupling
$\gamma = .1$

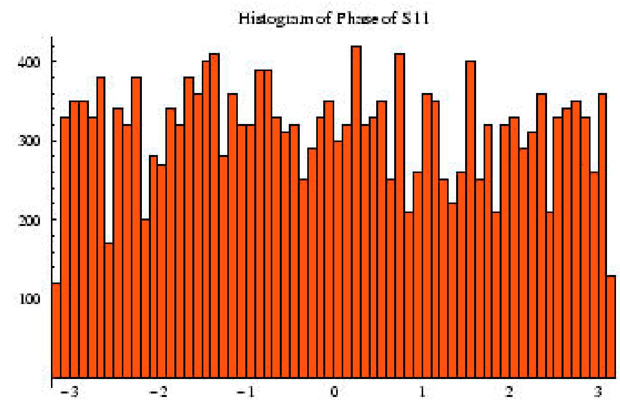
Optimal Coupling
$\gamma = 1/\pi$

Strong Coupling
$\gamma = 1.0$

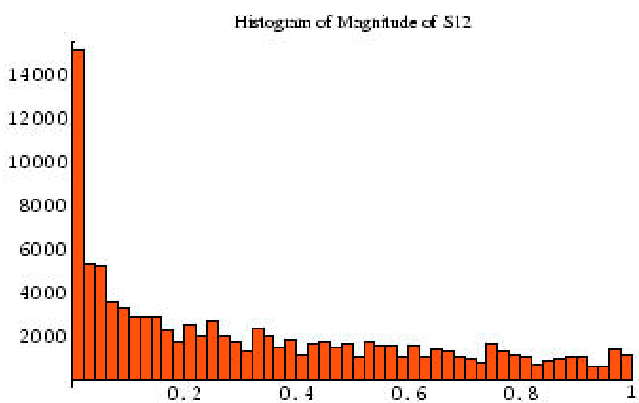
HFSS

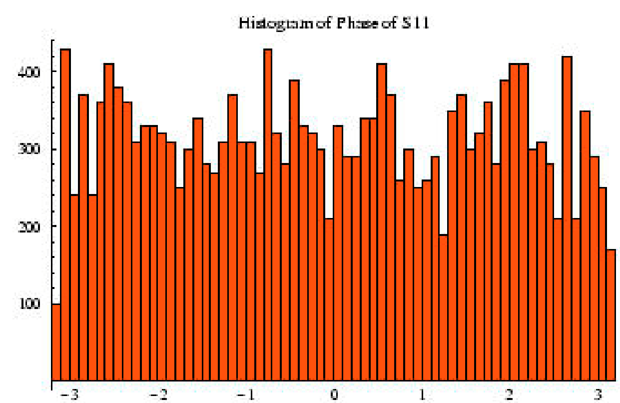
Statistics of S_{ij} - Optimal coupling, $\gamma=1/\pi$





S_{11}



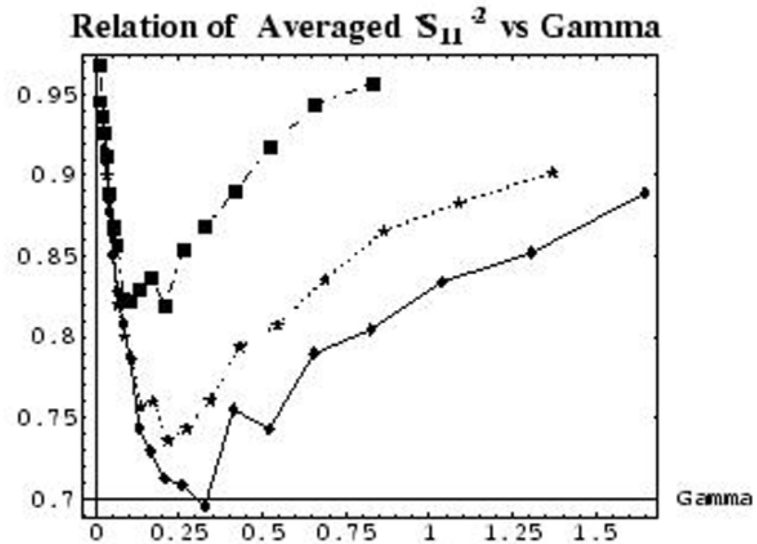


S_{12}

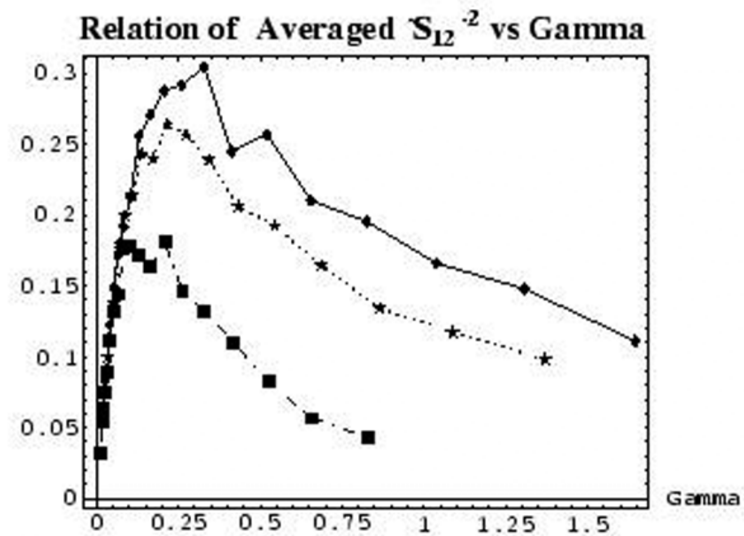
$|S_{ij}|$

Arg $[S_{ij}]$

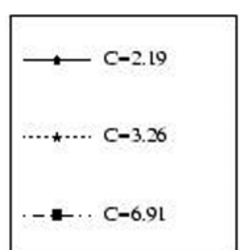
Average of S_{ij} vs Coupling



S_{11}



S_{12}



$$\ln\left(\frac{k_{max}^2 - k^2}{k^2}\right) = C$$

Future Directions

- Direct comparison of random coupling model with
  - -random matrix theory
  - -HFSS solutions
  - -Experiment

- Exploration of increasing number of coupling channels

- Role of Scars on low period orbits

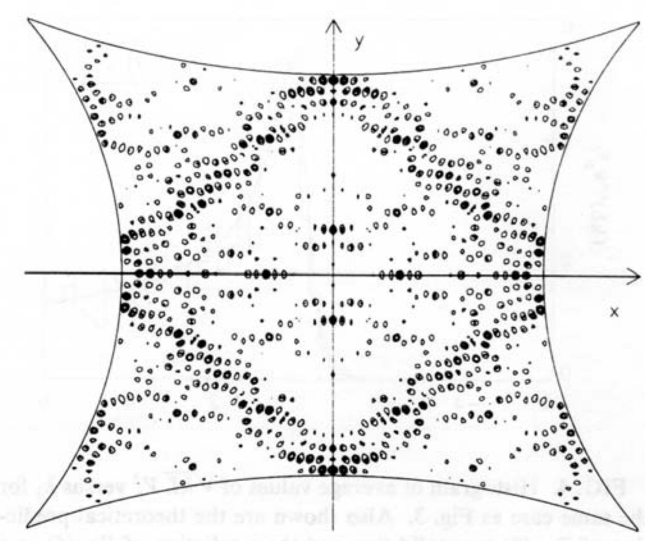
- Generalize to systems consisting of circuits and fields

Role of Scars?

- Eigenfunctions that do not satisfy random plane wave assumption

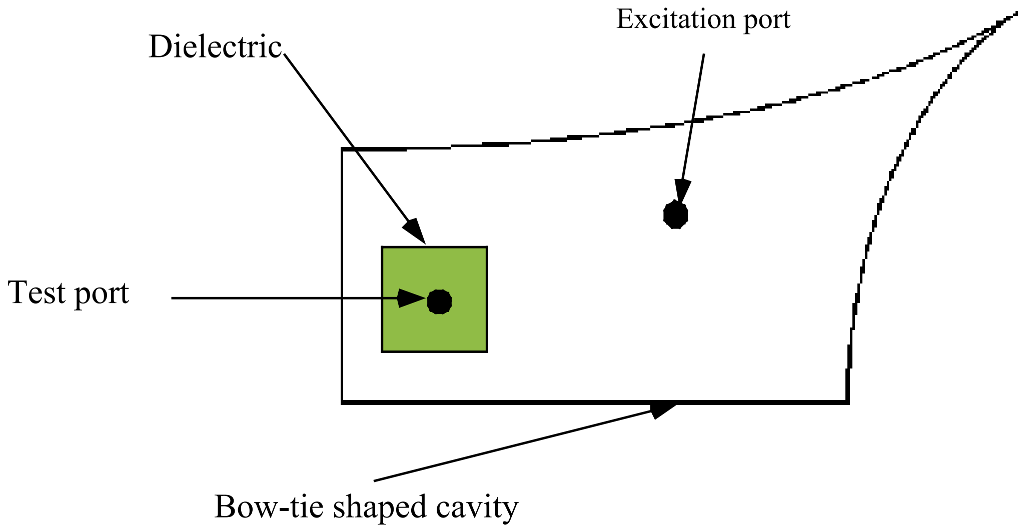
- Scars are not treated by either ramdom matrix or chaotic eigenfunction theory

- Semi-classical methods
- coupled S matrices



Bow-Tie with diamond scar

Future directions

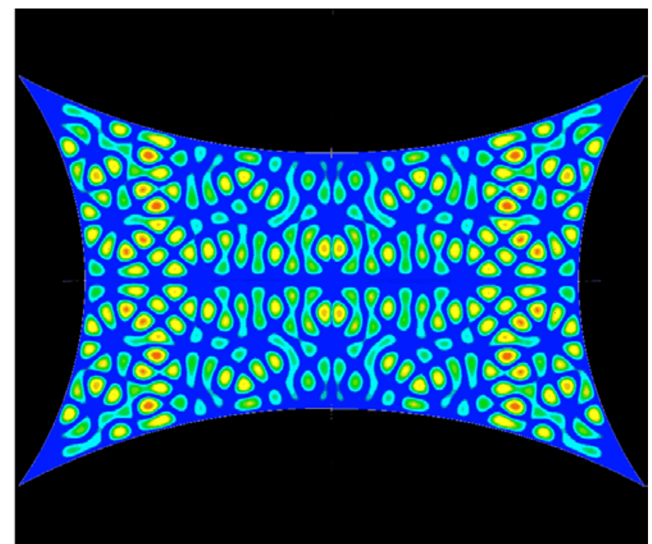


• Can be addressed

-theoretically

-numerically

-experimentally



HFSS simulation courtesy J. Rodgers

Features:

Ray splitting

Losses

Additional complications to be added later