Shear flow generation by drift waves revisited

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The generation of shear flow by drift waves is an important area of investigation. In a recent paper [Chen *et al.*, Phys. Plasmas **7**, 3129 (2000)] an elegant formalism was developed for the generation of shear/zonal flows by drift waves in a toroidal plasma. The study of shear flow generation in fluids and plasmas is put into perspective. In this paper it is shown how a simple slab geometry analysis can lead to a similar dispersion relation and highlight the subtle differences between the slab and toroidal geometries. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1340618]

I. INTRODUCTION

The observation of the low to high confinement (L–H) transition in tokamaks heralded significant optimism for achieving the goal of fusion in a laboratory plasma.[1–3] More recently, the enhanced confined modes observed in all the major devices has also has added to the optimism.[4,5] As a consequence, understanding of this remarkable self-organization of the tokamak plasma in these good confinement regimes is an important area of research. Central to all these enhanced confinement regimes, be they in the edge region or the core region of the plasma, is the generation of sheared flows or zonal flows, which are believed to be responsible for suppressing fluctuations and inhibiting transport. Thus, it is important to understand the mechanisms for the generation of shear flow.

We first put into historical perspective the study of shear flow generation in fluids and plasmas. It is only in the recent past that the linear theory of the generation of shear flow has been investigated. The first study that identified the shear flow instability is the work of Howard and Krishnamurty,[6] where the authors showed that tilting of the fluid vortices in a Rayleigh–Bénard convection experiment at low Reynolds number was due to the onset of the shear-flow instability. In the context of plasma physics, Drake *et al.*[7] and Finn *et al.*[8] developed a four-wave model for the instability, which leads to the generation of shear flow for driven vortices for a fluid represented by the incompressible Navier–Stokes equation in two dimensions. This study was motivated by the observation of shear-flow generation in the three-dimensional simulation of drift-resistive ballooning modes in the edge region of a tokamak by Guzdar *et al.*[9] Although Diamond and Kim[10] recognized earlier that the Reynolds stress was capable of driving shear flow, the actual instability mechanism that was a generalized parametric instability, was only understood due to the studies cited above.

As shown in the earlier work,[6–9] the basic shear-flow instability is a parametric instability involving a pump wave (which could be any normal mode like the drift, ion temperature gradient, resistive ballooning modes to name a few), the shear flow and a single sideband. The fourth mode used in the four-wave model was a higher spatial harmonic of the shear flow and was necessary to conserve the average vorticity. This led to a modification by Hermiz *et al.*[11] of the truncated model used by Howard and Krishnamurty,[6] since the latter model did not conserve average vorticity. Later on, using the Hasegawa–Mima–Charney equation, Guzdar[12] studied the shear-flow generation by Drift/Rossby waves. In this study it was shown that the pump wave coupled to the shear flow, its higher harmonic, as well as two drift-wave sidebands. In the context of parametric processes the instability due to coupling to the two sidebands is generically a modulational instability. The finite frequency associated with the drift/Rossby wave was responsible for introducing the second sideband. More recently, Chen *et al.*[13] have performed an elegant analysis of the shear/zonal flow generation by drift and/or ion temperature gradient (ITG) driven modes in a toroidal plasma using the coupling of a pump drift/ITG mode to the shear/zonal flow and two sidebands. These authors used a kinetic description of the plasma. Finally, recent work by Jenko *et al.*[14] and Dorland *et al.*[15] using a Kelvin–Helmholtz instability analysis for periodic shear flows, showed that due to the difference in the adiabatic response of electrons for the ITG modes and the adiabatic response of the ions for the electron temperature gradient (ETG) modes, the shear flow growth rates are significantly different for these two modes. This has major implications in the nonlinear saturation and transport scaling for these modes.

A more recent development for understanding the zonal flow as an instability is the work of Lebedev *et al.*,[16] Kaw *et al.*,[17] and Smolyakov *et al.*,[18] where the drift waves are represented by a wave-kinetic equation coupled to the zonal flow equation. For this case, the modulational instability analysis for a broad spectrum of drift waves yields a growth rate that depends on the square root of the amplitude of the pump. However recently, for a monochromatic wave packet for the drift wave, Smolyakov *et al.*[19] have shown that like the coherent modulational instability analysis, the growth rate is proportional to the amplitude of the pump. The latter is a stronger instability. In using the wave-packet formulation one tacitly assumes that the drift waves have a much smaller spatial scale compared to the scale of the shear flow. This separation of scale is not necessary in the coherent wave formulation. In retrospect, the earlier modulational instability studies by Sagdeev *et al.*[20] and Shapiro *et al.*,[21] using the wave packet formulation for the interaction of short-scale-length drift waves with long wavelength convective cells can be viewed as a precursor to the recent work, since the shear flow is a special case of the convective cell.

Here we provide a simple slab analysis of the electrostatic drift-wave zonal flow interaction model and recover the basic results derived in the recent work of Chen *et al.*[13] for the toroidal case. We show the differences between the slab and toroidal calculation, and also extend the regime of validity of the analysis to high mode numbers. The present theory gives an explicit mode number dependence of the maximally growing zonal flow.

II. BASIC EQUATIONS

The basic equations for the study of drift mode–zonal flow interaction are

$$\frac{\partial}{\partial t}n+\nabla\cdot(n\mathbf{v}_\perp)=0, \tag{1}$$

with

$$\mathbf{v}_\perp=-\frac{c}{B}\nabla\phi\times\hat{z}-\frac{c}{\Omega_i B}\frac{d}{dt}\nabla\phi, \tag{2}$$

with

$$\frac{d}{dt}=\frac{\partial}{\partial t}-\frac{c}{B}\nabla\phi\times\hat{z}\cdot\nabla \tag{3}$$

and

$$\frac{n}{n_0}-1=\frac{e}{T_e}[\phi-\overline{\phi}]. \tag{4}$$

Here $n=n_0(x)+\delta n_d$ is the total density, with n_0 the equilibrium density, with a spatial dependence, and δn_d the density associated with the drift waves. Here T_e is the electron temperature, $\Omega_i=eB/m_ic$ the ion gyrofrequency, e the positive magnitude of the electron charge, m_i the ion mass, c is the speed of light. Equation (4) is the correct adiabatic response for the electrons, as first pointed out by Dorland and Hammett[22] and more recently by Chen *et al.*[13] In deriving the velocity as the sum of the $E\times B$ and the polarization drift we have ordered the time scale associated with the drift wave (and zonal flows) to be ordered smaller than the ion gyrofrequency, i.e., $(d/dt)/\Omega_i\sim\delta$, where $\delta\ll 1$ is a smallness parameter. We also assume that $\delta n_d/n_0(0)\sim(\nabla_\perp L_n)^{-1}\sim\delta$. Here L_n is the scale length of the equilibrium density inhomogeneity. With these orderings, writing $\phi=\overline{\phi}+\delta\phi_d$, normalizing the potential to T_e/e and density n to $n_0(0)$, we substitute the velocity and density into the continuity equation, Eq. (1), and obtain the set of equations that describes the coupling of the drift waves to the zonal/shear flow,

$$(1-\rho_s^2\nabla^2)\frac{\partial}{\partial t}\phi_d-\frac{c_s^2}{\Omega_i}\nabla\phi_d\times\hat{z}\cdot\nabla\ln n_0-\frac{c_s^2}{\Omega_i}\nabla\langle\phi\rangle\times\hat{z}\cdot\nabla\phi_d+\frac{c_s^2\rho_s^2}{\Omega_i}\nabla\cdot[\nabla\langle\phi\rangle\times\hat{z}\cdot\nabla\nabla\phi_d+\nabla\phi_d\times\hat{z}\cdot\nabla\nabla\langle\phi\rangle]=0, \tag{5}$$

where $c_s=\sqrt{T_e/m_i}$, $\rho_s=c_s/\Omega_i$, $\phi_d=e\,\delta\phi_d/T_e$, and $\langle\phi\rangle=e\overline{\phi}/T_e$, and

$$\frac{\partial}{\partial t}\nabla^2\langle\phi\rangle-\frac{c_s^2}{\Omega_i}\nabla\cdot[\nabla\phi_d\times\hat{z}\cdot\nabla\phi_d]=0. \tag{6}$$

These sets of equations are different from the Hasegawa–Mima–Charney (HMC) model, which describes the drift-wave/drift-wave interaction but misses the lowest order drift-wave/zonal interaction [the third term on the left-hand side of Eq. (5)]. This is because in deriving the Hasegawa–Mima–Charney equation one has assumed that the electron is dominated by the adiabatic approximation given by Eq. (4) *without* the $\overline{\phi}$ piece. As a consequence, the third term in our current equation (5) does not appear in the HMC model. Also for the shear flow, the adiabatic limit is incorrect and the dynamics is strictly two dimensional like the incompressible Navier–Stokes fluid.

III. LOW-DIMENSIONAL MODEL

We first develop a low-dimensional model for the generation of shear/zonal flow by assuming that

$$\phi_d=\phi_{d0}\cos k_y(y-Ct)+\phi_{ds}\sin k_x x\,\sin k_y(y-Ct)+\phi_{dc}\sin k_x x\,\cos k_y(y-Ct), \tag{7}$$

with $C=V/(1+k_y^2\rho_s^2)$, $V=-(c_s^2/\Omega_i)d\ln n_0/dx$, and

$$\langle\phi\rangle=\phi_z\cos k_x x. \tag{8}$$

The interesting aspect of this truncated model is that the drift wave consists of the pump wave ϕ_{d0}, which couples to the two sidebands ϕ_{ds} and ϕ_{dc}, as was done in the earlier work.[12] In the more recent work by Chen *et al.*,[13] instead of using the sine and cosine sidebands, the complex wave representation for the two sidebands is used. In the parlance of mode coupling processes, the instability obtained for coupling of the pump wave to the two sidebands by the shear/zonal flow is referred to as a modulational instability. The equivalence of the two approaches has been well documented for the modulation instability of large-amplitude electromagnetic waves in the work of Kaw *et al.*[23] and Gurevich.[24] Using this representation, the coupled nonlinear equation for the four waves are

$$\frac{d}{dt}\phi_{d0}-\frac{c_s^2}{2\Omega_i}k_xk_y\phi_z\phi_{ds}=0, \tag{9}$$

$$(1+k_\perp^2\rho_s^2)\frac{d}{dt}\phi_{ds}+\frac{k_yVk_x^2\rho_s^2}{(1+k_y^2\rho_s^2)}\phi_{dc}+\frac{c_s^2}{\Omega_i}k_xk_y(1+k_y^2\rho_s^2-k_x^2\rho_s^2)\phi_z\phi_{d0}=0, \tag{10}$$

$$(1+k_\perp^2\rho_s^2)\frac{d}{dt}\phi_{dc}-\frac{k_y V k_x^2\rho_s^2}{(1+k_y^2\rho_s^2)}\phi_{ds}=0, \tag{11}$$

$$\frac{d}{dt}\phi_z+\frac{c_s^2}{2\Omega_i}k_x k_y \phi_{d0}\phi_{ds}=0. \tag{12}$$

These equations are an extension of the nonlinear equations derived by Chen *et al.*[13] to include finite ion Larmor radius effects (with electron temperature) on the drift waves obtained from the polarization drift term. If we assume that ϕ_d, ϕ_{dc}, $\phi_z \ll \phi_{d0}$, and that the perturbed state variables have a time dependence given as $\exp(\gamma t)$, then the linearized system of equations (10)–(12) yield the following dispersion relation:

$$\gamma^2=\frac{c_s^4 k_x^2 k_y^2}{\Omega_i^2(1+k_\perp^2\rho_s^2)^2}\left(\frac{(1+k_y^2\rho_s^2-k_x^2\rho_s^2)(1+k_\perp^2\rho_s^2)}{2}\phi_{d0}^2 - \frac{V^2 k_x^2\rho_s^2}{c_s^2(1+k_y^2\rho_s^2)^2}\right). \tag{13}$$

Here $k_\perp^2=k_x^2+k_y^2$. Also, this dispersion relation is generically similar to one derived by Guzdar[12] using the HMC equation. The differences arise because of the limitations of the HMC equation in describing the drift wave/zonal flow interaction as well as the finite size of the geometry in the x direction in the earlier work.[12] This dispersion relation is the slab version of Eq. (10) in the work of Chen *et al.*[13] and also includes the large mode-number limit because of the terms associated with the nonlinear polarization drift in the drift-wave components. If these terms were neglected, then the above dispersion relation would reduce to that in Ref. 13 with one significant difference, which is due to the toroidal geometry. For the toroidal plasma, the zonal/shear flow is dominated by the trapped ions. Thus Eq. (12) for the shear flow in toroidal geometry gets replaced by

$$\epsilon^{1/2}\Delta_p^2\frac{d}{dt}\phi_z+\frac{c_s^2}{2\Omega_i}k_x k_y\rho_s^2\phi_{d0}\phi_{ds}=0. \tag{14}$$

Here $\epsilon=r/R$ is the inverse aspect ratio and Δ_p is the banana width of the trapped ions. For trapped ions, the banana width is the characteristic scale size of the excursion. The first term is the effective polarization drift for the $\epsilon^{1/2}$ fraction of trapped ions (compared to the total ion population). If we recall that $\Delta_p=\sqrt{\epsilon}\rho_p$, with ρ_p the poloidal Larmor radius, we basically recover (within a multiplicative factor 1.6) the inertia term on the left hand side of Eq. (3) of Chen *et al.*[13] The factor 1.6 arises from a more detailed analysis by Rosenbluth and Hinton.[25] Thus, in the toroidal case the dispersion relation should be

$$\gamma^2=\frac{c_s^4 k_x^2 k_y^2}{\Omega_i^2(1+k_\perp^2\rho_s^2)^2}\left[\left(\frac{\rho_s^2}{1.6\epsilon^{1/2}\Delta_p^2}\right)\times\frac{(1+k_y^2\rho_s^2-k_x^2\rho_s^2)(1+k_\perp^2\rho_s^2)}{2}\phi_{d0}^2-\frac{V^2k_x^2\rho_s^2}{c_s^2(1+k_y^2\rho_s^2)^2}\right]. \tag{15}$$

The inertia of the trapped ions reduces the growth rate by decreasing the destabilization term [the first term in the square bracket on the right-hand side (rhs)].

This dispersion relation shows that modes with $k_x^2\rho_s^2 > 1+k_y^2\rho_s^2$ are always stable. The threshold for the amplitude of the pump wave is due to the detuning or the frequency difference between the pump and the two sidebands. By setting the growth rate to zero, the critical k_x below which the instability occurs can be determined. If $k_x^2\rho_s^2 \ll 1$, then the critical wave number k_{xc} below which the drift wave is unstable to the generation of shear flow (in the slab case) is

$$k_{xc}\rho_s=\frac{1}{\sqrt{2}}(1+k_y^2\rho_s^2)^2\frac{\phi_{d0}c_s}{V}. \tag{16}$$

For the toroidal case, this critical wave number would be smaller by a factor of $\rho_s/(\sqrt{1.6}\epsilon^{1/4}\Delta_p)$. This condition would be valid for small amplitude waves for which $e\delta\phi_{d0}/T_e \ll \rho_s/L_n$. However, if $e\delta\phi_{d0}/T_e \gg \rho_s/L_n$, then the critical wave number $k_{xc}\rho_s=(1+k_y^2\rho_s^2)^{1/2}$, beyond which no instability occurs.

IV. CONCLUSIONS

We have derived a low-dimensional system of equations for the generation of shear/zonal flow by drift waves. The linear stability analysis in the slab geometry of this simple system displays the similarities and yet subtle differences with the recent toroidal analysis of Chen *et al.*[13] The evolution equation [Eq. (14)] for the shear flow in a toroidal plasma is due to the trapped ions, while in a slab the shear flow equation is due to the circulating ions [Eq. (12)]. As a consequence the growth rates in the toroidal case are smaller [Eq. (15)] compared to the slab case [Eq. (13)]. We have also extended the regime of validity into larger k_x wave number space (comparable and larger than ρ_s^{-1}) to provide a short-wavelength cutoff for the zonal flow instability.

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