Steady-state magnetohydrodynamic plasma flow past conducting sphere

Z. Chacko and A. B. Hassam
Department of Physics, University of Maryland, College Park, Maryland 20742

(Received 17 December 1996; accepted 30 April 1997)

An analytic solution to the problem of strongly magnetized plasma flow past a smooth, conducting sphere is considered. The magnetic field is taken to be uniform at very large distances and the sphere is assumed to be unmagnetized. In addition, the flow speed is assumed to be subsonic and super-Alfvénic. It is shown that a steady state solution is possible only if the frozen-in condition can be relaxed near the surface of the sphere. By inclusion of a small resistivity, the presence of two, nested boundary layers near the surface is demonstrated. The magnetic field is shown to drape about the sphere with a scale size of the order of the square root of the resistivity. © *1997 American Institute of Physics.* [S1070-664X(97)01108-7]

I. INTRODUCTION

The problem of magnetized plasma flow past conducting obstacles is of primary interest in space plasma physics. Flow of the solar wind past planets, flow of interstellar winds past stars, or the motion of satellites and tethers through magnetospheric plasmas are the obvious examples.[1,2] Laboratory terella experiments also fall into this category.[3]

The flowing plasma is invariably magnetized, while the obstacles themselves could be magnetized (the Earth) or unmagnetized[4] (Venus and satellites). In the case of large obstacles, ideal magnetohydrodynamics is an appropriate starting point for calculations. The steady-state problem is the first step.

Generally speaking, the problem is difficult because it is inherently three-dimensional (3D). In particular, consider that the flow vector of the external wind defines one direction for the problem: if both the obstacle and the flowing plasma are unmagnetized, then the problem may have azimuthal symmetry about the flow axis; if, however, either the obstacle or the flowing plasma are magnetized, then the problem is generally three-dimensional since the flow axis, the magnetic dipole axis, and the direction of the magnetic field in the flowing plasma do not, in general, have any particular relation to each other. The problem is made even more difficult because the equations are nonlinear.

Given these difficulties, the usual approach to this problem is numerical simulation. Indeed, much has been learned from this approach.[5] An analytical solution, even in asymptotic limits, is, however, still desirable as it would illuminate better some aspects of the problem. In this paper, we present an analytic solution to the problem of steady-state magnetized flow past a conducting sphere. To make the problem tractable, we make the following simplifying assumptions: (1) the sphere is assumed to be unmagnetized and smooth; and (2) the relevant energy densities in the problem, the internal energy, the flow kinetic energy, and the magnetic energy, are assumed to satisfy the inequalities

$$3p/2 \gg nMu^2/2 \gg B^2/2. \qquad (1)$$

The latter assumptions mean that the flow is subsonic and super-Alfvénic. Finally, the magnetic field at infinity is assumed to be uniform. With these assumptions, a tractable analytic solution is obtained.

The analytic solution we obtain is richly textured, both from physical as well as mathematical viewpoints, and exhibits several interesting features, even though the solution is obtained in a restricted domain. In particular, three interesting features that emerge from our calculation are as follows.

(1) We show that a steady-state solution does not exist within the context of ideal magnetohydrodynamics (ideal MHD). In particular, finite resistivity has to be introduced to obtain a steady state. The resistivity introduces a boundary layer around the sphere. This boundary layer is quite intricate in that there are actually *two* boundary layers: one nested inside the other. The magnetic field drapes about the sphere. The draping is very large for those field lines heading toward the leading edge of the sphere—the lines affected most would be the ones that have an ''impact parameter'' less than a scale size that is of order a times the square root of the inverse magnetic Reynolds number, where a is the radius of the sphere.

This non-ideal nature of our restricted problem illuminates an important aspect of the full problem of flow past a magnetized object. The reason for a non-ideal component to our solution has to do with the stagnation points of the flow pattern that reside at the leading and the trailing edges of the sphere. Because of the flow stagnation, the frozen-in flux piles up at the leading edge. To reach a steady state, resistivity is necessary to relax the frozen-in condition and to allow the flux to slip away from the stagnation region. Now, we emphasize that this inclusion of resistivity is necessary in spite of the fact that the sphere is unmagnetized. It is well known from the magnetospheric work that in the case of an embedded planetary dipole field, X points appear around the planet. The general belief is that the X points lead to magnetic reconnection, which, as is well known, must involve a breakdown in frozen in by, say, resistivity. The solution to our problem shows that resistivity is necessary even in the absence of X points.

(2) We show that the problem is reducible from being three dimensional, in, say, spherical coordinates (r,θ,ϕ), to being two dimensional, in r and θ only. To be sure, this reducibility has to do with the fact that we have assumed that

the magnetic field is relatively weak. Nonetheless, it is interesting to see that by assuming that variables depend on ϕ only as $\exp(+i\phi)$, we can obtain a consistent solution that is entirely two dimensional otherwise. This result constitutes a powerful insight when it comes to addressing this problem by numerical means: for, if one desired to solve the time-dependent MHD equations numerically for this problem, at least for where the energy densities were ordered appropriately, one could possibly reduce the difficulty of the problem by using the first harmonic in ϕ ansatz, *a priori*, thus rendering the equations two-dimensional.

(3) The above features as well as other general features of this problem, such as the draping behavior of the field, the appearance of the tail, and the widths of the boundary layers, can be deduced to hold quite generally not only for spheres but for any obstacles that possess azimuthal symmetry.

To return to our introductory remarks, the starting point of our calculation would be the set of magnetohydrodynamic (MHD) equations. As we have discussed, we reduce the difficulty of the problem by adopting the hierarchy $B^2/2 \ll nMu^2/2 \ll 3p/2$. Since the system is subsonic, the pressure is approximately constant and the reduced MHD equations can be written as

$$\nabla\times(nM\mathbf{u}\cdot\nabla\mathbf{u}) = \nabla\times(\mathbf{j}\times\mathbf{B}), \tag{2}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3}$$

$$\nabla\times\mathbf{E} = 0, \tag{4}$$

$$\mathbf{E} \equiv -\mathbf{u}\times\mathbf{B} + \eta\mathbf{j}, \tag{5}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{6}$$

$$\mathbf{j} \equiv \nabla\times\mathbf{B}. \tag{7}$$

Standard notation is used. Now, the kinetic energy term in (2) is larger than the $\mathbf{j}\times\mathbf{B}$ term. Thus, the latter can be neglected. The resulting equation is satisfied if the condition,

$$\nabla\times\mathbf{u} = 0, \tag{8}$$

is met. From (3) and (8), $\mathbf{u}$ can be determined. Once $\mathbf{u}$ is known, $\mathbf{B}$ is obtained by using

$$\mathbf{u}\times\mathbf{B} - \eta\nabla\times\mathbf{B} = \nabla\Phi, \tag{9}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{10}$$

Equation (9) is obtained from (4) where Φ is the electric potential, i.e., $\mathbf{E} = -\nabla\Phi$. The equation,

$$\mathbf{u}\cdot\nabla\Phi = \eta\,\nabla^2\Phi, \tag{11}$$

follows directly from (9) by taking the $\mathbf{u}$ component and using (8).

In what follows, we begin by solving for Φ from (11) and then use (9) and (10) to find $\mathbf{B}$. The paper is organized as follows. In the next section, we solve for the electric potential. In doing so, we show that two nested boundary layers, occurring near the surface of the sphere, are necessary in order to obtain a solution to the problem that is well behaved everywhere. In Sec. III, we derive the equations for the magnetic field and show that the field undergoes a large ''draping'' around the sphere. We discuss our findings in Sec. IV.

For convenience, we use normalized units in this paper. All length scales are normalized to the radius of the sphere, a, and all velocities are normalized to the flow speed at infinity, u_0. The electric and magnetic fields at infinity are related according to $cE_0/B_0 = u_0$. Thus, in units of u_0, $B_0 = cE_0$. The fundamental parameter that governs the complicated boundary layer structure in this problem is the magnetic Reynolds number, au_0/η: in normalized units, this number is $1/\eta$.

II. THE ELECTRIC FIELD

Consider a subsonic plasma flow past a smooth sphere of radius $a = 1$. As discussed, the flow velocity is obtained from $\nabla\cdot\mathbf{u} = 0$ and $\nabla\times\mathbf{u} = 0$. In spherical coordinates, where the flow at infinity is in the positive z direction, the solution is

$$\mathbf{u} = (1 - 1/r^3)\cos\,\theta\hat{e}_r - (1 + 1/2r^3)\sin\,\theta\hat{e}_\theta. \tag{12}$$

The flow can also be written in terms of the potentials λ and χ, representing, respectively, the streamlines and the equipotentials of $\mathbf{u}$. The potential surfaces are orthogonal to each other and will be useful later on. They are defined according to

$$\mathbf{u} = -\nabla\chi = -\nabla\phi\times\nabla\lambda^2/2, \tag{13}$$

$$\lambda = (r^2 - 1/r)^{1/2}\,\sin\,\theta, \tag{14}$$

$$\chi = -(r + 1/2r^2)\cos\,\theta. \tag{15}$$

The electric and magnetic fields are to be obtained from the equations

$$\mathbf{u}\times\mathbf{B} - \eta(\nabla\times\mathbf{B}) = \nabla\Phi, \tag{16}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{17}$$

where the electric field $\mathbf{E}$ is given by $\mathbf{E} = -\nabla\Phi$. The potential Φ satisfies

$$\eta\,\nabla^2\Phi = \mathbf{u}\cdot\nabla\Phi. \tag{18}$$

We begin by solving Eq. (18) to obtain Φ. The boundary conditions are that (i) the surface of the sphere is an equipotential and (ii) at large distances from the obstacle the field returns to an asymptotic value. Assuming the electric field initially points along the positive x axis, we demand

$$\Phi(r\rightarrow\infty) = -E_0 r\,\sin\,\theta\,\cos\,\phi. \tag{19}$$

Equation (18) is difficult to solve exactly. However, since η is small, we expect the left-hand side to be negligible everywhere, except perhaps in a boundary layer. We thus first find the external solution ($\eta = 0$) from the equation $\mathbf{u}\cdot\nabla\Phi = 0$. This solution, consistent with boundary condition (ii), is simply

$$\Phi_{\text{ext}} = -E_0\lambda\,\cos\,\phi, \tag{20}$$

where λ is defined above. Fortuitously, Φ_{ext} also satisfies boundary condition (i). However, the relevant physical quantity, the electric field $\mathbf{E} = -\nabla\Phi$, diverges as $(r-1)^{-1/2}$. Thus, the external solution, obtained for $\eta = 0$, is clearly unacceptable: a boundary layer as $r\rightarrow 1$ is indicated.

To find the thickness of the boundary layer we substitute Φ_{ext} for Φ on the left-hand side (LHS) of Eq. (18) and find that for $(r-1)\approx O(\eta^{1/2})$, the terms arising from the radial derivatives on the LHS are comparable to the terms on the right-hand side (RHS). Hence the outer solution appears to be valid outside a boundary layer of thickness $(r-1)\approx O(\eta^{1/2})$ around the sphere.

Now to obtain the solution in the neighborhood of the boundary layer we return to the full equation (18) and reduce it as follows. We assume that the radial derivatives dominate, i.e., $r\partial/\partial r \gg \partial/\partial\theta$. Thus, the LHS of (18) can be approximated as $\partial^2\Phi/\partial r^2$. On the RHS, however, both u_r and u_θ have to be retained since, although $u_\theta \gg u_r$, the sharper radial derivative makes $u_\theta\partial_\theta \sim r u_r \partial/\partial r$. Thus, the boundary layer equation for Φ is given by

$$\eta\left(\frac{\partial^2\Phi}{\partial r^2}\right)_\theta \approx \mathbf{u}\cdot\nabla\Phi. \tag{21}$$

To make progress, we now switch to a new set of independent variables, the streamline variable, λ, introduced in Eq. (14), and θ. These are more natural to the problem since the RHS is the just the derivative along the streamline. More explicitly, the RHS in these coordinates becomes

$$\mathbf{u}\cdot\nabla = (1/r)(1+1/2r^3)\sin\,\theta(\partial/\partial\theta). \tag{22}$$

The LHS, on making the approximation that derivatives in λ dominate over derivatives in θ, becomes a second-order operator in $\partial/\partial\lambda$. Making the further approximation $(r-1)\ll 1$, Eq. (21) reduces to

$$\eta\frac{1}{\lambda}\,\frac{\partial}{\partial\lambda}\left(\frac{1}{\lambda}\,\frac{\partial\Phi}{\partial\lambda}\right)_\theta = -\frac{2}{3}\,\frac{1}{\sin^3\theta}\left(\frac{\partial\Phi}{\partial\theta}\right)_\lambda. \tag{23}$$

In this form, we note that the equation has the structure of a diffusion equation with diffusion coefficient $\eta\sin^3\theta$, θ being the time part and λ^2 being the space part. We want a solution that goes over to $-E_0\lambda\cos\phi$ for $\lambda \gg O(\eta^{1/4})$. Hence we make the ansatz

$$\Phi = -E_0\lambda\cos\phi f(\beta), \tag{24}$$

where

$$\beta = \tau(\theta)/\lambda^4, \tag{25}$$

$$\tau(\theta) = -6\eta\int_{\theta_0}^{\theta}\sin^3\theta\, d\theta. \tag{26}$$

The choice for $\tau(\theta)$ is obvious from the structure of the ''time'' part of the equation. The choice of β as the governing independent variable is based on the similarity structure of the above diffusion equation. Substituting for Φ in Eq. (23), we find an ordinary differential equation for $f(\beta)$:

$$\beta^2 f'' + (\beta - 1/4)f' - 1/16 f = 0. \tag{27}$$

Equation (27) determines the detailed boundary layer structure. We extract the solution in the following manner. We make no attempt to reduce this equation to a standard form. Rather, we determine the asymptotic behaviors of the equation as $\beta\to 0$ and $\beta\to\infty$, then determine which asymptotic behaviors at either end are consistent with our desired boundary and matching conditions, and then confirm, via standard numerical ''shooting'' methods, that one of the solutions to the above equation does indeed ''connect'' between the desired asymptotic limits.

To elaborate, let us first establish the asymptotic behavior. As $\beta\to 0$, there is a regular as well as a singular solution. These asymptotic solutions are readily found to be

$$f_{0,1}\to 1, \tag{28}$$

$$f_{0,2}\to \int^\beta d\beta'\,\exp(-1/4\beta'). \tag{29}$$

For $\beta\to\infty$, the equation is equidimensional. The asymptotic behaviors are

$$f_{\infty,1}\to \beta^{-1/4}, \tag{30}$$

$$f_{\infty,2}\to \beta^{+1/4}. \tag{31}$$

It can now be readily confirmed numerically that the solution that behaves as $f_{0,1}$ as $\beta\to 0$ connects to the solution that behaves as $f_{\infty,1}$ as $\beta\to\infty$. We will label this solution as $f_1(\beta)$. Then, f_1 is the solution we require since it matches onto the outer solution for large $(r-1)$ and satisfies the inner boundary condition.

To summarize, we have now identified a boundary layer as $r\to 1$. The solution in the layer is given by Eq. (24). The external solution is given by Eq. (20). The layer solution matches on to the external solution as $f(\beta)\to 1$. The latter occurs as $\beta\to 0$. The variable β becomes small when λ is large, i.e., as we go out in the streamlines, as we would expect. The thickness of the layer is given by $\lambda\sim O(\eta^{1/4})$.

On closer examination, however, the layer thickness exhibits an interesting behavior. In particular, the choice of the angle θ_0 is important in determining the shape of the layer. To elaborate, we note from (25) that as $\theta\to\theta_0$, $\tau(\theta)\to 0$. Thus, to asymptotically match onto the external solution, λ would have to exceed $\eta^{1/4}$ for all θ except near θ_0, where λ need not be that large. That is to say, the layer thickness is reduced near θ_0.

The correct choice for our problem is $\theta_0=\pi$. Consider the condition $\beta\ll 1$ for $r\to 1$. The condition becomes $\tau\ll(r-1)^2\sin\theta$. When θ is neither near 0 nor near π, τ is of order η and, thus, the thickness of the layer is given by $\eta^{1/2}\ll(r-1)$. If θ is near π, τ is of order $\eta(\pi-\theta)^4$ and the condition $\beta\ll 1$ is still $\eta^{1/2}\ll(r-1)$ since the $(\pi-\theta)$ terms cancel from both sides. If, however, θ is near 0, τ is of order η and the condition $\beta\ll 1$ becomes $\eta^{1/2}/\theta^2\ll(r-1)$: the layer becomes much thicker as θ approaches zero.

There is thus an asymmetry in the boundary layer thickness, with an infinitely long tail being drawn out at the trailing edge, a feature to be expected in flow-past-obstacle problems. This finding, however, leads to another concern. If the boundary layer is drawn out in a tail, we may have an inconsistency, in that the reasoning leading up to the identification of the tail, namely Eq. (23) and its solution, were obtained under the assumption that $(r-1)\ll 1$. On closer examination, however, we can show that there is no real inconsistency, as follows.

Let us reexamine Eq. (23) near $\theta\to 0$. Since $\tau\sim\eta$ and $\lambda\sim(r^2-1/r)^{1/2}\theta$, the LHS of this equation is negligible if θ is small and $r\sim 1$. Thus, in this region, the RHS is domi-

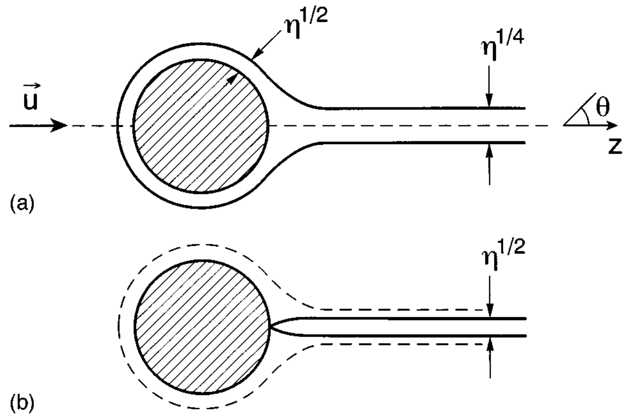

FIG. 1. (a) The shape of the outer boundary layer. (b) The structure of the inner and outer boundary layers.

nant. Now, upon examining the RHS operator, given in Eq. (22), we see that the condition that the RHS be dominant is unaffected by whether or not r is close to 1 [observe that there are no $(r-1)$ factors in this operator]. A recognition of this fact leads to the conclusion that the boundary layer equation (23) is operative not only in the $r\rightarrow 1$ domain, but is also operative for all r. It follows that our solution (24) is also valid for finite r and our identification of the tail, as described above, is correct.

We thus conclude that the solution to Φ, as given by Eq. (24), is valid in a boundary layer that hugs the sphere and then stretches out to infinity in a tail along the trailing edge. The layer structure is depicted in Fig. 1(a). We recall, however, that the entire analysis of this boundary layer solution assumes that variations along θ are slower than variations along r. This assumption should be checked for self-consistency, particularly in the tail region where $r\sim O(1)$. We note also that we have used (λ,θ) coordinates to obtain the previous boundary layer solution with the ansatz $\partial/\partial\lambda \gg \partial/\partial\theta$ being made to obtain the boundary layer equation (23). Clearly, as $\theta\rightarrow 0$, the surfaces of constant λ and constant θ become almost degenerate. This also suggests that a closer examination of the assumptions made is in order. We therefore scrutinize the boundary layer solution inside the long tail, but as $\theta\rightarrow 0$. In doing so, we find, upon demanding self-consistency, the presence of a new boundary layer. Specifically, we resurrect the $\partial/\partial\theta$ terms that were discarded on the LHS of Eq. (21) and demand, for the solution given by Eq. (24), that these terms be self-consistently small compared with one of the terms in the $\mathbf{u}\cdot\nabla\Phi$ operator on the RHS. From this exercise, we can show that self-consistency is obtained only if $\lambda >> O(\eta^{1/2})$. Since in the region of small θ the boundary layer has the form $\lambda\sim O(\eta^{1/4})$, i.e., $\theta\sim O(\eta^{1/4})$, the angular terms become relevant in a region completely enclosed by this boundary layer. This confirms that a new, nested boundary layer exists within the boundary layer previously found.

To find the solution to Φ in this new boundary layer we return to Eq. (18). We now make the ansatz that all radial derivatives, $\partial/\partial r$, are small compared to the angular derivatives, $r^{-1}\partial/\partial\theta$. (This assumption, which is consistent with our knowledge of the outer boundary layer solution within this region, will be reexamined for self-consistency later.) With these assumptions, Eq. (18) reduces, in the limit $\theta\rightarrow 0$, to

$$\frac{\eta}{r^2}\left[\left(\frac{\partial^2\Phi}{\partial\theta^2}\right)_r+\frac{1}{\theta}\left(\frac{\partial\Phi}{\partial\theta}\right)_r-\frac{\Phi}{\theta^2}\right]=(1-1/r^3)\left(\frac{\partial\Phi}{\partial r}\right)_\theta-(1+1/2r^3)\frac{\theta}{r}\left(\frac{\partial\Phi}{\partial\theta}\right)_r. \tag{32}$$

Choosing as new independent variables λ and r, the equation becomes

$$\frac{1}{\eta}\left(\frac{\partial\Phi}{\partial r}\right)_\lambda=\frac{1}{\lambda}\frac{\partial}{\partial\lambda}\left(\lambda\frac{\partial\Phi}{\partial\lambda}\right)_r-\frac{\Phi}{\lambda^2}. \tag{33}$$

Once again, the boundary layer equation takes the form of a diffusion equation with, in this case, r being the time coordinate and λ still being the space coordinate. Since the new boundary layer is nested well within the previous boundary layer, the outer limit of the solution for the inner boundary layer will match on to the inner limit of the outer boundary layer. From Eqs. (24) and (27), the inner limit of the outer boundary layer solution is given by

$$\Phi\rightarrow\kappa\lambda^2\cos\phi, \tag{34}$$

where κ is a constant that can be determined only by solving Eq. (27) for f exactly. We now assume that in the nested boundary layer the solution for Φ has the form

$$\Phi=\kappa\lambda^2 g(t), \tag{35}$$

where

$$t=s(r)/\lambda^2,$$

and the functional forms of g and $s(r)$ are to be determined. Inserting this into (33), we may show that the appropriate choice for $s(r)$ is

$$s(r)=\eta(r-r_0), \tag{36}$$

where r_0 is a constant to be determined. In this case, the equation for g is given by

$$t^2g''-(t+1/4)g'+(3/4)g=0. \tag{37}$$

As before, we solve this equation by examining the asymptotic behaviors and confirming, numerically, that one of the solutions does indeed ''connect'' the two desired asymptotic behaviors. The asymptotic behavior of (37) is as follows. For $t\rightarrow 0$,

$$g_{0,1}\rightarrow 1, \tag{38}$$

$$g_{0,2}\rightarrow\int^t dt'\,\exp(-1/4t'). \tag{39}$$

For $t\rightarrow\infty$,

$$g_{\infty,1}\rightarrow t^{1/2}, \tag{40}$$

$$g_{\infty,2}\rightarrow t^{3/2}. \tag{41}$$

In order to satisfy the inner boundary condition, we desire the behavior $g\rightarrow t^{1/2}$ as $t\rightarrow\infty$. We confirm numerically that $g_{\infty,1}$ indeed connects to $g_{0,1}$, label the corresponding solution g_1, and choose this as our desired solution. Hence our solution does indeed match onto the inner form of the outer solution for $\theta \gg \eta^{1/2}$ and satisfy the boundary condition on the sphere.

We now seek to determine the constant r_0. It is easy to verify that the choice of $r_0 = a = 1$ ensures that the nested boundary layer has the form $\theta \approx O(\eta^{1/2})$, even close to the sphere. This is because $t = \eta s/\lambda^2 \simeq \eta/(3\sin^2\theta)$ close to the sphere, since in this region $\lambda^2 \approx 3(r-1)\sin^2\theta$. This choice also ensures that the assumption made earlier, that the radial derivatives be negligible within the second boundary layer, is self-consistent, as may be verified by calculating them from the expression for Φ given as $\Phi = \kappa\lambda^2 g(t)$.

This completes the determination of the electric field. The complete boundary layer structure is shown in Figs. 1(a) and 1(b). Figure 1(a) depicts the outer boundary layer; Fig. 1(b) shows the inner boundary layer, nested within the outer one. Outside both the boundary layers, for $\lambda >> O(\eta^{1/4})$, the solution is given by Eq. (20). In the outer boundary layer, defined as $O(\eta^{1/2}) \ll \lambda \sim O(\eta^{1/4})$, Φ is given by Eq. (24) with $f_1(\beta)$ being used. For the inner boundary layer, defined as $O(\eta^{1/2}) \sim \lambda \ll O(\eta^{1/4})$, Φ is given by Eq. (35) with $g_1(t)$ being used.

III. THE MAGNETIC FIELD

Having solved for the electric field we now proceed to determine the magnetic field. The relevant equations are, once again,

$$\mathbf{u}\times\mathbf{B} - \eta(\nabla\times\mathbf{B}) = \nabla\Phi \tag{42}$$

and

$$\nabla\cdot\mathbf{B} = 0, \tag{43}$$

where Φ is now known. In what follows we take advantage of the fact that $\nabla\lambda$, $\nabla\phi$, and $\mathbf{u}$ form an orthogonal coordinate system to expand the magnetic field as

$$\mathbf{B} = -F\ \nabla\lambda - G\ \nabla\phi - H\mathbf{u}, \tag{44}$$

where F, G, and H are scalar functions to be determined. We recall that $\mathbf{u}$ itself can be written as the gradient of a potential, i.e., $\mathbf{u} = -\nabla\chi$, where χ is given in Eq. (15). In this event, Eq. (42) can be decomposed into a set of three equations coupling F, G, and H,

$$F + \eta\frac{\partial H}{\partial\lambda} + \eta\frac{\partial F}{\partial\chi} = -\frac{\partial\Phi}{\partial\phi}\frac{\lambda}{\rho^2 u^2}, \tag{45}$$

$$G + \eta\frac{\partial H}{\partial\phi} + \eta\frac{\partial G}{\partial\chi} = \frac{\partial\Phi}{\partial\lambda}\frac{\rho^2}{\lambda}, \tag{46}$$

$$\eta\frac{\partial F}{\partial\phi} - \eta\frac{\partial G}{\partial\lambda} = \frac{\partial\Phi}{\partial\chi}\lambda. \tag{47}$$

Similarly, Eq. (43) is transformed to

$$\frac{\partial H}{\partial\chi}u^2 = \nabla\cdot(F\ \nabla\lambda) + \frac{\partial G}{\partial\phi}\frac{1}{\rho^2}. \tag{48}$$

Here, $\rho = r\sin\theta$. In Eqs. (45)–(48), the functions F, G, and H are functions of the coordinates (λ,χ,ϕ), and the partial derivatives are taken with respect to these coordinates. The set of three equations (45)–(47) is not independent; any one can be obtained from the other two and the defining equation for Φ, Eq. (18).

Having already calculated Φ, we know it has the form $\Phi = \Phi_0(\lambda,\chi)\cos\phi$. Examining Eqs. (45)–(48), we find that the ansatz,

$$F = F_0(\lambda,\chi)\sin\phi, \tag{49}$$

$$G = G_0(\lambda,\chi)\cos\phi, \tag{50}$$

$$H = H_0(\lambda,\chi)\sin\phi, \tag{51}$$

is self-consistent. Using this, (45)–(47) reduce to

$$F_0 + \eta\left(\frac{\partial H_0}{\partial\lambda}\right)_\chi + \eta\left(\frac{\partial F_0}{\partial\chi}\right)_\lambda = \frac{\Phi_0\lambda}{\rho^2 u^2}, \tag{52}$$

$$G_0 + \eta H_0 + \eta\left(\frac{\partial G_0}{\partial\chi}\right)_\lambda = \left(\frac{\partial\Phi_0}{\partial\lambda}\right)_\chi\left(\frac{\rho^2}{\lambda}\right), \tag{53}$$

$$-\eta F_0 + \eta\left(\frac{\partial G_0}{\partial\lambda}\right)_\chi = -\left(\frac{\partial\Phi_0}{\partial\chi}\right)_\lambda\lambda, \tag{54}$$

while (48) becomes

$$u^2\left(\frac{\partial H_0}{\partial\chi}\right)_\lambda = \nabla\cdot(F_0\nabla\lambda) - \frac{G_0}{\rho^2}. \tag{55}$$

Now in order to solve this set of equations, we substitute from (53) and (54) into (55) and use the identity $\lambda|\nabla\lambda| = \rho u$ to obtain, for G_0, the equation

$$\eta\ \nabla^2 G_0 - \mathbf{u}\cdot\nabla G_0 = \nabla\phi\cdot(\nabla\Phi_0\times\nabla\rho^2), \tag{56}$$

where, henceforth ∇^2 is taken to mean

$$\nabla^2 \equiv \nabla_\perp^2 - \frac{1}{\rho^2}, \tag{57}$$

$$\nabla_\perp^2 \equiv |\nabla\lambda|^2\left(\frac{\partial^2}{\partial\lambda^2}\right)_\chi + \nabla^2\lambda\left(\frac{\partial}{\partial\lambda}\right)_\chi + u^2\left(\frac{\partial^2}{\partial\chi^2}\right)_\lambda. \tag{58}$$

Equation (56) above is identical to the equation for Φ_0 solved earlier, except that it is inhomogeneous. However, since the homogeneous operator is exactly the same in both equations we expect that the solution of the inhomogeneous equation will exhibit the same boundary layer structure as the solution of the homogeneous equation found earlier. We will take this as an *ansatz* whose self-consistency can be verified from the solution of the equation.

The outer solution for G_0 can be read off from Eq. (53) in the $\eta\to 0$ limit as

$$G_0 = -E_0\rho^2/\lambda. \tag{59}$$

To determine the solution in the outer boundary layer, we simplify the master equation (56) by keeping only those terms from the homogeneous differential operator which we know to be relevant in this region. We then have

$$\frac{1}{4\lambda}\frac{\partial}{\partial\lambda}\left(\frac{1}{\lambda}\frac{\partial G_0}{\partial\lambda}\right)_\theta - \left(\frac{\partial G_0}{\partial\tau}\right)_\lambda = -\left(\frac{\partial\rho^2}{\partial\tau}\right)_\lambda\frac{1}{\lambda}\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_\theta, \tag{60}$$

where we have assumed that derivatives in the λ direction dominate over those in the θ direction. We have also assumed that $r\to 1$. To make progress, we note that the expression $\partial\Phi_0/\partial\lambda$ is just a function of the variable $\beta=\tau/\lambda^4$, introduced earlier: this fact can be checked from Eqs. (24) and (25). Then, from a dimensional examination of the terms in Eq. (60), the structure of the outer solution (59), and an expectation that the same boundary layer structure as before will be operative, we make the substitution for G_0,

$$G_0=-E_0\rho^2 S(\beta)/\lambda. \tag{61}$$

The resulting equation is

$$\frac{1}{4}\frac{\partial}{\partial\lambda}\left[\frac{1}{\lambda}\frac{\partial}{\partial\lambda}\left(\frac{S}{\lambda}\right)\right]_\theta-\left(\frac{\partial S}{\partial\tau}\right)_\lambda=\frac{1}{\rho^2}\left(\frac{\partial\rho^2}{\partial\tau}\right)_\lambda\left[S+\frac{1}{E_0}\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_\theta\right]. \tag{62}$$

In arriving at (62), we have made the additional approximation that the quantity ρ^2 is very slowly varying over the scale of $O(\eta^{1/4})$. We now claim that the solution to Eq. (62) can be obtained by separately setting to zero both the LHS and the RHS of this equation. Namely, we first let

$$S(\beta)=-\frac{1}{E_0}\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_\theta. \tag{63}$$

We note, from Eq. (24), that

$$\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_\theta=-E_0(f_1-4\beta f_1'). \tag{64}$$

Thus, Eq. (63) represents a solution for S if and only if the LHS of (62) is identically zero if we insert for $S(\beta)$ the expression on the RHS of Eq. (64). This proposition can be proven as follows. First, the LHS of (62) in terms of S becomes

$$16\beta^2 S''+(32\beta-4)S'+3S=0. \tag{65}$$

This is in itself remarkable, in that Eq. (62) has been reduced to an ordinary differential equation. We now insert for S the expression $(f_1-4\beta f_1')$ to obtain the differential equation,

$$64\beta^3 f_1'''+(240\beta^2-16\beta)f_1''+(108\beta-12)f_1'-3f_1=0. \tag{66}$$

It is straightforward to show that upon using the governing differential equation for f_1, Eq. (27), Eq. (66) is identically satisfied.

We thus conclude that the solution to G_0 in the first boundary layer is

$$G_0=(\rho^2/\lambda)(\partial\Phi_0/\partial\lambda)_\theta, \tag{67}$$

where Φ_0 is given in Eq. (24). To be sure, this solution for G_0 was obtained from Eq. (60), wherein we assumed that $r\to 1$. We recall that the extent of the first boundary layer includes a tail wherein r is indeed of $O(1)$. Thus, as in the previous section when we obtained Φ in the first boundary layer, we can ask about the validity of solution (67) in the tail region of the first boundary layer. The answer is that expression (67) is valid also inside this tail. The reason for this is directly analogous to the corresponding discussion in the previous section, viz. the discussion preceding Eq. (32): the boundary layer equation for G_0, Eq. (60), is valid *throughout* the first boundary layer. In particular, the $\partial/\partial\lambda$ terms in (60) are small in the tail region and the size and structure of the other two terms of that equation do not depend on factors proportional to $(r-1)$. The reasoning is in direct analogy with the corresponding reasoning in the previous section.

We now turn our attention to the inner boundary layer. Once again we keep only those terms in the master equation, which we know to be relevant from the solution of the homogeneous equation. The master equation in this region then simplifies to

$$\eta\frac{1}{\lambda}\frac{\partial}{\partial\lambda}\left(\frac{\lambda\partial G_0}{\partial\lambda}\right)_r-\eta\frac{G_0}{\lambda^2}=\left(\frac{\partial G_0}{\partial r}\right)_\lambda-\frac{\partial}{\partial r}\left(\frac{\rho^2}{\lambda}\right)_\lambda\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_r+\frac{2r^2}{\alpha^2}\left(\frac{\partial\Phi_0}{\partial r}\right)_\lambda, \tag{68}$$

where we emphasize that the expressions for Φ_0 to be used in this equation are those from the corresponding, i.e., the inner, boundary layer [see Eq. (35)]. The quantity $\alpha=(r^2-1/r)^{1/2}$. Now the solution for G_0 from the outer boundary layer,

$$G_0=-E_0\rho^2(f_1-4\beta f_1')/\lambda, \tag{69}$$

has the inner form (as $\beta\to\infty$)

$$G_0\to 2\kappa\rho^2. \tag{70}$$

As in the outer boundary layer we attempt a solution of the type

$$G_0=2\kappa\rho^2 T(t), \tag{71}$$

where $s=r-1$ and $t=\eta s/\lambda^2$, as used earlier in Eq. (36). This form of the solution preserves the boundary layer structure of the homogeneous equation. The resulting equation is

$$\eta\frac{1}{\lambda}\frac{\partial}{\partial\lambda}\left(\frac{\lambda\partial T}{\partial\lambda}\right)_r-\eta\frac{T}{\lambda^2}-\left(\frac{\partial T}{\partial r}\right)_\lambda-\frac{1}{\kappa\lambda^2}\left(\frac{\partial\Phi_0}{\partial r}\right)_\lambda=\frac{1}{\rho^2}\left(\frac{\partial\rho^2}{\partial r}\right)_\lambda\left[T-\frac{1}{2\kappa\lambda}\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_r\right]. \tag{72}$$

As before, we have made the approximation that ρ^2 is very slowly varying over the narrow inner layer. Again, as before, the solution to Eq. (72) can be obtained by separately setting to zero both the LHS and the RHS of this equation. Namely, we first let

$$2\kappa T(t)=\frac{1}{\lambda}\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_r. \tag{73}$$

We note, from Eq. (35), that

$$\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_r=2\kappa\lambda(g-tg'). \tag{74}$$

Note that this expression depends only on the variable t. Thus, Eq. (73) represents a solution for T if and only if the LHS of (72) is identically zero if we insert for $T(t)$ the expression on the RHS of Eq. (74). This proposition can be proven as follows. First, the LHS of (72) in terms of g, using (35), becomes

$$4t^2T''-(4t+1)T'+3T=g'. \tag{75}$$

As before, Eq. (72) has been reduced to an ordinary differential equation. We now insert for T the expressions (73) and (74) to obtain the differential equation

$$-4t^3g'''+tg''-(3t+1)g'+3g=0. \tag{76}$$

It is straightforward to show upon using the governing differential equation for g, Eq. (37), that Eq. (76) is identically satisfied. Hence, the solution to G_0 in the inner boundary layer is

$$G_0=(\rho^2/\lambda)(\partial\Phi_0/\partial\lambda)_r, \tag{77}$$

where Φ_0 is given in Eq. (35).

This completes the determination of G_0. Having done this it is relatively straightforward to obtain an expression for F_0 from Eqs. (45)–(47). The result is

$$F_0=\left(\frac{\partial G_0}{\partial\lambda}\right)_\phi-\frac{\partial}{\partial\lambda}\left[\left(\frac{\partial\Phi_0}{\partial\lambda}\right)_\phi\frac{\rho^2}{\lambda}\right]+\frac{\Phi_0\lambda}{u^2\rho^2}-\lambda\left(\frac{\partial^2\Phi_0}{\partial\chi^2}\right)_\lambda. \tag{78}$$

It is straightforward to verify that F_0 is zero on the surface of the sphere. This is actually the inner boundary condition on the equation for G_0 solved earlier.

Obtaining an expression for H, however, is not simple. It cannot simply be read off from Eqs. (45)–(47) because the solutions obtained for F_0 and G_0 are only approximate to order η. Thus, to calculate H_0 we return to Eq. (55), viz.

$$u^2\left(\frac{\partial H_0}{\partial\chi}\right)_\lambda=F_0\ \nabla^2\lambda+\left(\frac{\partial F_0}{\partial\lambda}\right)_\chi(\nabla\lambda)^2-\frac{G_0}{\rho^2}. \tag{79}$$

To evaluate H_0 we must integrate the above expression with respect to χ along contours of constant λ up to the point (λ,χ), where the value of H_0 is desired. Note that very close to the sphere the integration will traverse several boundary layers, rendering the calculation rather tedious. In view of this, we restrict our attention to the exterior region, i.e., outside of the boundary layers.

In the latter case, the expression for H_0 simplifies to

$$H_0=\int^\chi d\chi\left(\frac{\partial}{\partial\lambda}\frac{1}{u^2}\right)_\chi. \tag{80}$$

Note that the integrand is symmetric about the line $\theta=\pi/2$ on a given λ streamline. Thus, the integral need only be evaluated from $\theta=\pi$ to $\theta=\pi/2$, with the contribution from greater angles deduced from the angles less than $\pi/2$. In spite of these reductions, the integral is still difficult to evaluate. We therefore restrict our attention to two special cases, with the limited intent of extracting some qualitative information about the nature of H_0. The first case is when we are very far from the sphere, while the second is when we approach the sphere but stay outside the boundary layers.

Very far away from the sphere, $\lambda\gg1$, we have

$$\left(\frac{\partial u^2}{\partial\lambda}\right)_\chi\approx\left(\frac{1}{r^4}\right)(12-15\sin^2\theta)\sin\theta, \tag{81}$$

and $u^2\approx1$, $\lambda\approx r\sin\theta$, $\chi\approx-r\cos\theta$, with $r\approx(\chi^2+\lambda^2)^{1/2}$. In this case, H_0 is obtained to be

$$H_0\approx-(3/\lambda^3)A(-\cot\theta), \tag{82}$$

where A is defined by

$$A(x)=\int^x\left(\frac{3}{(1+t^2)^{5/2}}-\frac{5}{(1+t^2)^{1/2}}\right)dt. \tag{83}$$

Our solution implies that for contours that are always distant from the sphere, along a given angle θ the value of H decreases as the cube of the ''impact parameter'' λ.

We now approach the sphere but stay outside of the boundary layer. That is, we examine H_0 for λ such that $1\gg\lambda\gg O(\eta^{1/4})$. The required integral can be conveniently split into two regions: we first integrate up to a point (r_0,θ) such that $a\gg r_0-1\gg\lambda$ and subsequently from r_0 to $(\lambda,\pi/2)$. In this case, it is readily shown that the contribution from the first region is $O(\lambda^2)$ smaller than that from the second. Since our primary objective is to estimate the component of the magnetic field downstream from the sphere, we only evaluate the integral in the second region. In this case, r runs from r_0 to λ along the same λ contour. Since in this entire region we are close to the sphere, i.e., $r-1\ll1$, we have

$$u^2\simeq9[(\tfrac{1}{4})\sin^2\theta+(r-1)^2], \tag{84}$$

$$(\partial u^2/\partial\lambda)_\chi\simeq(27/4)(\lambda/u^2)(6t-\sin^2\theta), \tag{85}$$

where $t=r-1$. We first consider the integration over $dt|_\lambda$. Using the scaling $t\lambda^{-2/3}=T$, we obtain

$$\Delta H=-\frac{243}{2\lambda}\int_\infty^T\frac{dT|_\lambda T^2}{(9T^2+3/4T)^3}\simeq O(1/\lambda). \tag{86}$$

For the integral over $d\theta|_\lambda$, we use the scaling $\sigma=\sin\theta(1/\lambda)^{2/3}$ to obtain

$$\Delta H_2=\frac{81}{4\lambda}\int_0^1\frac{d\sigma}{\sigma}\frac{1}{(1/\sigma^4+9/4\sigma^2)^3}\simeq O(1/\lambda). \tag{87}$$

From this we draw the conclusion that close to the sphere but outside the boundary layers the component of the field parallel to the flow is large and of order $1/\lambda$. The same is true of any point downstream from the sphere on the same λ contour.

IV. SUMMARY AND DISCUSSION

This paper has been concerned with obtaining an analytic solution to the problem of magnetized (MHD) plasma flow past a smooth, unmagnetized, conducting sphere. To keep the problem tractable, we made several simplifying assumptions, notably assuming that the flow speeds were subsonic but super-Alfvénic. We also utilized an expansion in large magnetic Reynolds number. In spite of these assumptions, the calculation is quite involved, as we have seen. Two boundary layers were encountered and relatively complicated diffusion-type equations had to be solved. One can reasonably conclude that the next level of sophistication in this general problem area, allowing stronger magnetic fields, or considering magnetized spheres, for example, is likely to involve even more complexity.

The simplifications notwithstanding, several features were appreciated in the course of addressing this problem. As already discussed in the Introduction, we have shown that the steady-state problem is only well posed if resistivity is included. We conclude therefore that boundary layers are unavoidable in the general problem. It is interesting to speculate whether the problem of the magnetized sphere would have extended boundary layers, along the lines we found, in addition to the expected localized boundary layers in the vicinity of the X point. We also found, again as discussed in the Introduction, that by choosing appropriate harmonic behavior for the variables, $\cos\phi$ and $\sin\phi$ in our case, the problem could be reduced to being two dimensional. While this is a consequence of the weak field assumption, it nonetheless opens up an avenue for doing relatively quick, exploratory numerical studies in two-dimensions of an essentially 3D problem.

It is useful to consider why a boundary layer structure, of the particular form obtained, emerges. As already broached in the Introduction, the appearance of a boundary layer is to be expected because of the existence of stationary points in the flow pattern: the flux pile-up from the frozen-in theorem at these points can only be relaxed by resistivity. While this explains why there is a boundary layer close to the sphere, it is not clear why the layer stretches out into a tail. Some understanding of this may be obtained by examining the behavior of the external solution for H, the component of the magnetic field along the direction of the flow. We have seen by explicit calculation that H becomes very large as we approach the sphere. From the expression for H, Eqs. (86) and (87), we see that this component is equally large at points downstream from the sphere along the same λ contour. More important, we note from Eq. (51) that H varies with ϕ as $\sin\phi$. This implies that the magnetic field, while almost parallel to the flow in direction, reverses sign as we traverse the magnetotail. That is, the value of H changes discontinuously, from large and positive to large and negative across the downstream midplane. This is a clear indication that the boundary layer structure must extend well away downstream from the sphere and stretch into a long tail, to resolve the discontinuity in H. The facts that H becomes large in the tail region and that it has a $\sin\phi$ dependence are both completely general features of any problem that exhibits azimuthal symmetry in the flow pattern: thus, one might expect the boundary layer structure to be a general feature of problems of flow past an obstacle, at least for obstacles that exhibit azimuthal symmetry but possibly the case in general.

Finally, the weak field assumption, made throughout our analysis, needs reexamination. To begin with, we have found that the draping of the field is so strong that the field pile-up builds up in strength so as to possibly invalidate the weak field assumption. The latter would certainly be the case for very small resistivity. A closer examination of this question reveals the following. The problem as posed involves three small parameters: the inverse of the sonic Mach number, the Alfvén Mach number, M_A, and the inverse of the magnetic Reynolds number, $1/S$. Specifically, the latter two quantities are defined at infinity according to $M_A^2 \equiv (B^2/4\pi n M u^2)_\infty$ and $1/S \equiv (\eta/au)_\infty$. Now, the region of flow stagnation, occurring at the leading and trailing edges of the sphere, has two effects, both stemming from the frozen-in condition. Frozen-in results in a pile-up of the magnetic field at the stagnation points: this leads to a buildup of magnetic energy, or an increase in size of the local Alfvén Mach number, and it also means that resistivity must act to release the pile-up if steady state is to be reached. If $M_A \ll 1$, then the region in which the local Alfvén Mach number becomes finite is small compared with a. Likewise, if $1/S \ll 1$, the scale size in which the effective magnetic Reynolds number is of order unity is also small. This reasoning suggests that there should be a boundary layer associated with each smallness parameter. In fact, the particular problem addressed in the present paper is one where the boundary layer from M_A is of a much smaller scale than the $S^{-1/2}$ scale that we found from the resistive boundary layer. That is to say, for our problem, the resistivity is large enough so that the flow goes from super-Alfvénic to sub-Alfvénic only *inside* the outer boundary layer. Clearly, if M_A is not small enough for this hierarchy to hold, then the M_A boundary layer will occur first. That is to say, in the stagnation regions, one must include the $\mathbf{j}\times\mathbf{B}$ terms on par with the $nM\mathbf{u}\cdot\nabla\mathbf{u}$ terms. This is a different ordering than that assumed in this paper and is outside the scope of the present paper.

The identification of two effects both independently leading to boundary layers leads to the question of whether one or the other effect can be completely neglected. That is to say, can the buildup of the magnetic field be neglected if the resistivity is large enough; conversely, can the resistivity be neglected if the Alfvén Mach number is large enough? An examination of our problem shows that, in principle, both effects have to be resolved. For, even if the resistivity is large enough, the stagnation in the flow still necessarily occurs and, since the magnetic field drapes about the sphere, the field is strong enough inside the resistive boundary layer near the stagnation point so as to invalidate the weak field assumption. Thus, in principle, we would have to reexamine our boundary layer solution at the stagnation point and allow magnetic field forces to modify the potential flow. We have not addressed this in the present paper, although it is unlikely that the solutions would change dramatically as long as M_A were small enough. In the opposite limit, wherein the boundary layer from M_A occurs outside the resistive layer, we conclude that the resistivity is still essential. This is because, without resistivity, frozen in would still hold, and, in spite of the modification of the flow from magnetic forces, the stagnation point will still necessarily remain, leading to a time-secular field pile-up—resistivity would be essential to achieve a steady state. We therefore conclude that the effects of both M_A and $1/S$ are essential for a full description of the problem.

ACKNOWLEDGMENT

This work was supported in part by a grant from the National Science Foundation.

[1] J. R. Spreiter and A. Y. Alksne, in *Annual Reviews of Fluid Mechanics*, edited by M. D. Van Dyke (Annual Reviews, Palo Alto, CA, 1970), Vol. 2, pp. 313–354.
[2] D. B. Beard and F. S. Johnson, J. Geophys. Res. **66**, 4113 (1961).
[3] K. Schindler, Rev. Geophys. **7**, 51 (1969).
[4] J. R. Spreiter and S. S. Stahara, J. Geophys. Res. **85**, 7715 (1980).
[5] See, for example, J. A. Fedder and J. G. Lyon, Geophys. Res. Lett. **14**, 880 (1997).