k Spectrum of Finite Lifetime Passive Scalars in Lagrangian Chaotic Fluid Flows

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The power law exponent for the wave number power spectrum of a passive scalar field in Lagrangian chaotic flows is found to differ from the classical value of -1 (Batchelor's law) when the passive particles have a finite lifetime for exponential decay. A theory based on the chaotic dynamics of the passive scalar is developed and compared to numerical simulation results.

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The advection of passive scalars by fluid flows is of interest due to its relevance to many situations, such as industrial chemical mixing processes and various mixing phenomena of chemicals, pollutants, and temperature in the atmosphere and oceans. If a passive scalar is injected into a chaotic flow, the passive scalar field is advected, stretched, and folded due to the chaotic nature of the flow. This repeated stretching and folding results in the development of finer and finer structures in the passive scalar field until the fine scales are removed by diffusive smoothening. In particular, even in the absence of high wave number velocity field structures, it is known that a weakly diffusing passive scalar still shows the development of high wave number structures as long as the velocity field is *Lagrangian chaotic* [1,2]. In a *Lagrangian chaotic* flow, two adjacent fluid elements convected by the flow diverge exponentially in time. The wave number power spectrum has been widely used to characterize passive scalars. It is defined as

$$F_{\phi}(k,t) = 1/(2\pi)^D \int d\vec{k}'\, \bar{C}(\vec{k}',t)\delta(k - |\vec{k}'|), \quad (1)$$

which is the average of $\bar{C}(\vec{k},t)$ over all directions of the wave number vector $\vec{k}$ such that $|\vec{k}| = k$, where $\bar{C}(\vec{k},t)$ is the spatial Fourier transform of the two point correlation function $C(\vec{r},t) = \langle \phi(\vec{x}+\vec{r},t)\phi(\vec{x},t)\rangle$ (the angular bracket denotes spatial average). About forty years ago, Batchelor proposed a k^{-1} power law for the power spectrum of a passive scalar field that is continuously fed by a source at long wavelength [3]. Since then, many experimental and theoretical efforts (including numerical simulations) have been done to verify the validity for the k^{-1} law for the power spectrum [2,4–6].

In this Letter, we address passive scalar advection in two dimensional [$D = 2$ in Eq. (1)] *Lagrangian chaotic* flows, in the case where the passive scalar has a *finite lifetime* for exponential decay. (Batchelor's considerations apply to the case of infinite lifetime.) The case of finite lifetime is relevant in a variety of situations [7,8]. For instance, a passive scalar, which is advected by the velocity field in a system with a continuously injecting source and a continuously removing sink, will typically spend a finite time in the system. Other examples include the wave number power spectrum of the fluorescence of particles whose fluorescence intensity is decaying exponentially, or of the concentration of a dissociating AB chemical compound undergoing the reaction $AB \rightarrow A + B$. We find that the power spectrum of passive scalars with a finite lifetime still obeys a power law, but that the power law exponent is different from the -1 of Batchelor's k^{-1} law which is obtained for conserved passive scalars (infinite lifetime). This has been previously found in numerical work by Abraham [8] with reference to plankton distribution and temperature in the ocean. We show here how the power law exponent is determined by the dynamical properties of the chaotic flow and the lifetime of the passive scalar.

The passive scalar field $\phi(\vec{x},t)$ evolves according to the advection equation which we write in the form

$$\frac{\partial \phi(\vec{x},t)}{\partial t} + \vec{v}(\vec{x},t)\cdot\nabla\phi = \kappa\nabla^2\phi + S_{\phi}(\vec{x}) - D(\vec{x})\phi, \quad (2)$$

where $\vec{v}(\vec{x},t)$ is the fluid velocity field, κ is the diffusivity of the passive scalar, $S_{\phi}(\vec{x})$ is the source function of the passive scalar, and $D(\vec{x})\phi$ represents a loss term whereby the passive scalar field decays with time at the rate $D(\vec{x})$. Without the $D(\vec{x})\phi$ term, Eq. (2) is the conventional convection problem of a conserved passive scalar. In what follows, we will compare theoretical results with direct numerical computations of the passive scalar using a split step technique to solve Eq. (2). For the later purpose, we consider a two dimensional square domain, $[-\pi,\pi]\times[-\pi,\pi]$, with periodic boundary conditions, and for the source and sink we choose $S_{\phi}(x,y) = \sin x \sin y$ and $D(\vec{x}) = \frac{\gamma}{2}[2 - |\tanh\frac{x+\pi}{L_D}| - |\tanh\frac{x-\pi}{L_D}|]$ [9]. (The case $D = \text{const}$ has also been simulated with similar results.) For the velocity field $\vec{v}(\vec{x},t)$ we consider a particular two dimensional flow which we obtain by direct numerical solution of the Navier-Stokes equations with spatially distributed forcing. The conditions for the solution of the Navier-Stokes problem {viscosity, forcing, and a frictional force $[-\nu\vec{v}(\vec{x},t)]$} were chosen (see Ref. [6] for details) so as to obtain a flow that mimics features observed in the experiments of Williams *et al.* [4]. In particular, since the Reynolds number is not large, the wave number energy spectrum of the flow is

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concentrated at relatively long wavelength; that is, the power spectrum of the flow decays with increasing k to negligible values very much before the wave number region where the passive scalar spectrum is affected by diffusion (high Schmidt number, $S_c = 200$). For example, the energy density in the velocity field at $k = 10$ is about 10^{-4} of the energy density at $k = 1$. During the course of the simulation, one can find 3–5 vortices in the velocity field. These vortices slowly move around one another and are spontaneously generated and annihilated as time progresses.

The solid line in Fig. 1 shows the power spectrum of the passive scalar from the direct numerical simulation for parameter values $\kappa = 2.5 \times 10^{-5}$, $\gamma = 400$, $L_D = 2\pi/40$, and a grid size of 1024^2. The power spectrum is obtained by time-averaging instantaneous power spectra. Almost one decade of the scaling range in the power spectrum is observed. The dashed line represents the theory with $F(k) \sim k^{-(1+\xi)}$, where $\xi \simeq 0.5$ for this flow, while the diamonds represent the wave packet model (WPM), which will be described subsequently. Reasonable agreement between the theory and the direct numerical simulation is observed, although for this value of κ the scaling range is only about one decade. The dotted line is plotted in order to allow the comparison with Batchelor's k^{-1} law. (In the case without the sink, our numerical computation of (2) is consistent with an exponent of -1; see [6]).

Now, we discuss how we solve Eq. (2) using the eikonal-type wave packet model introduced by Antonsen *et al.* in Refs. [2,10]. A modification in the WPM equations for our problem is required from the WPM equations of the conserved passive scalar which is investigated in

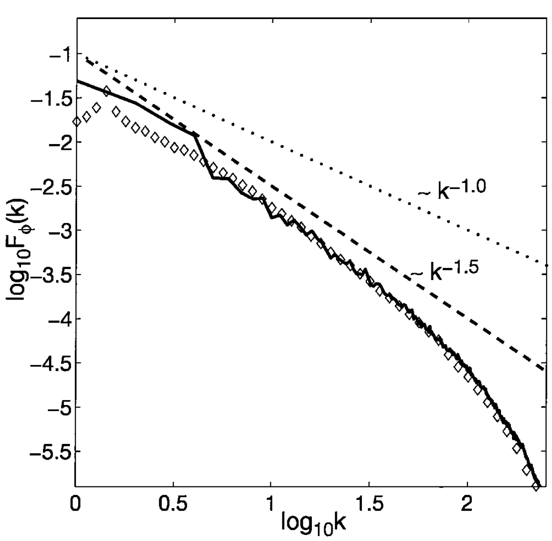

FIG. 1. Power spectrum of the passive scalar field. The solid line is from the direct numerical simulation. The diamonds are obtained using the wave packet model. The dashed line and the dotted line are from our theory and Batchelor's k^{-1} law, respectively.

[2,6]. In WPM, the passive scalar field is approximated by a superposition of wave packets, $\phi(\vec{x}, t) = \sum_j \phi_j(\vec{x}, t)$, where $\phi_j(\vec{x}, t)$ is in the form of a modulated sinusoidal function of $\vec{x}$ which is localized about $\vec{x} = \vec{x}_j$ with a characteristic wave vector $\vec{k}_j$. Each wave packet attached to the chaotic flow is governed by the following set of ODE's:

$$d\vec{x}_j/dt = \vec{v}(\vec{x}_j, t), \tag{3}$$

$$d\vec{k}_j/dt = -(\nabla\vec{v}) \cdot \vec{k}_j, \tag{4}$$

$$d\Omega_j/dt = -2\kappa k_j^2 \Omega_j - 2D(\vec{x}_j)\Omega_j, \tag{5}$$

where $\vec{x}_j$ and $\vec{k}_j$ are the location and the characteristic wave number of the jth wave packet, and $\Omega_j = 1/(2\pi)^2 \int d\vec{x} |\phi_j|^2$ is the variance of the wave packet. Because of the chaotic nature of the flow, the characteristic wave number of the wave packet increases exponentially as time progresses, $|\vec{k}| \sim \exp(ht)$. The first term in Eq. (5) describes the effect of diffusion and the second term is due to loss of the passive scalar (i.e., the finite lifetime). If the characteristic wave number of the wave packet is comparable to $\sim 1/\sqrt{\kappa}$ or the location of the wave packet is in the region where $D(\vec{x})$ is large, a strong decrease in the variance of the wave packet occurs. Under this model, the power spectrum of the passive scalar can be expressed as

$$F_\phi^{wpm}(k) = \sum_j \Omega_j[t = t_j(k)]\delta[k - |\vec{k}_j(t)|], \tag{6}$$

where the summation is over all wave packets, and $t_j(k)$ is the time when the jth wave packet wave number reaches k, and, in practice, we use (6) to form histogram approximations to F_ϕ.

To apply the wave packet method to our problem, we steadily inject new wave packets at low wave number in a random way over space and evolve them to higher wave number using (3)–(5). The initial wave vectors for the passive scalar wave packets are set to $\vec{k} = (1, 1)$. The results at higher k are insensitive to this choice. The diamonds in Fig. 1 show the power spectrum of the passive scalar obtained using the wave packet method. The same diffusivity as the direct numerical simulation case, $\kappa = 2.5 \times 10^{-5}$, is used. We find very good agreement between the results from the full numerical simulation and the results from the wave packet method. Better conformity to the theoretical expectation is expected if the diffusivity of the passive scalar is substantially reduced. Such a reduction is not practical for our full numerical simulation of (2), but is easily accommodated by wave packet computations, which involve solving ordinary (as opposed to partial) differential equations. The good agreement between the two computation methods seen in Fig. 1 lends support to this approach. Accordingly, in Fig. 2 we plot as a solid line the result for the case where a much smaller diffusivity coefficient $\kappa = 2.5 \times 10^{-9}$ is used for the wave packet model with the same velocity flow. A

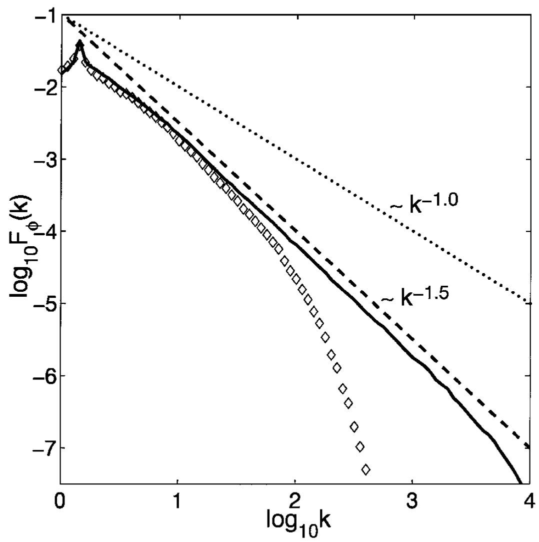

FIG. 2. Power spectrum of the passive scalar obtained using the wave packet model for different diffusivities. The diamonds are for $\kappa = 2.5 \times 10^{-5}$ and the solid line is for $\kappa = 2.5 \times 10^{-9}$ with the same chaotic fluid flow for both diffusivities.

clear power law behavior of the passive scalar spectrum is observed over more than two decades of the spectral range, and the power law exponent corresponds very well to the theoretical prediction.

To see the nature of the decay of the scalar induced by the loss term $D(\vec{x})\phi(\vec{x},t)$ in Eq. (2), we proceed as follows. We steadily inject many wave packets distributed randomly in space as described earlier and evolve their variances according to Eq. (5) with κ set equal to zero. Figure 3 shows a semilogarithmic plot of the decay of the total variance as a function of the time that the individual wave packets stay in the system. This decay is exponential for a long time, as evidenced by the linear behavior

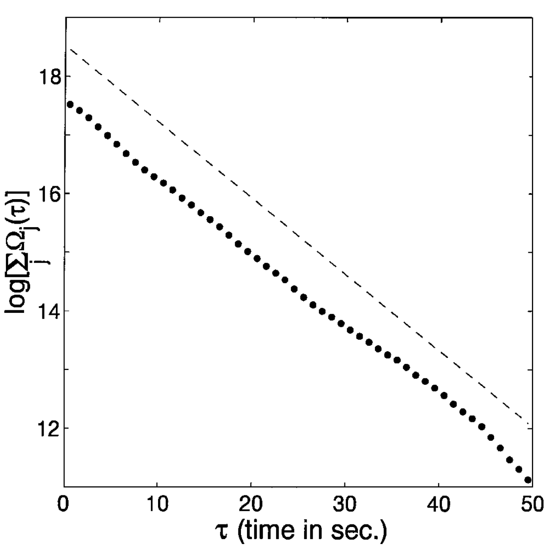

FIG. 3. Semilogarithmic plot of the sum of variances (dots) vs time. The dashed line is $\Omega \sim \exp[-\tau/7.69]$.

of the graph. The negative reciprocal of the slope of a straight line fit then defines the average decay time T. For our given flow, source, and sink, we obtain $T = 7.69$.

The power law exponent can be theoretically obtained in the following way. Let $\lambda_j = \log k_j/k_0$ denote the exponentiation experienced by wave packet j. From Eq. (6), by averaging over wave packets with a fixed exponentiation, we get

$$\tilde{F}_\phi(k) = k^{-1}\langle\Omega(\tau)\rangle_\lambda\,, \tag{7}$$

where $\tau_j = t_j(k) - t_j(k_0)$ is the time difference between the time when the jth wave packet is launched, $t_j(k_0)$, and the time when its wave number reaches k, $t_j(k)$ (or, equivalently, the time which is required for a wave packet to have a fixed exponentiation of λ). Because of the exponentially decaying property of the variances of the wave packets, we can express (7) as $\tilde{F}_\phi(k) \sim k^{-1}\langle e^{-\tau/T}\rangle_\lambda$ or

$$\tilde{F}_\phi(k) \sim k^{-1}\int dh\, Q(h|\lambda)\exp\!\left(-\frac{\lambda}{hT}\right), \tag{8}$$

where T is the average lifetime of the passive scalar. $Q(h|\lambda)$ is the conditional distribution of the finite-exponentiation Lyapunov exponent $h = \lambda/\tau$ at fixed λ [10], and it is assumed in (8) that there is no correlation between the lifetime of an individual wave packet and its finite-time Lyapunov exponent h. This will be verified later. [From (8), we see that $\tilde{F}_\phi \sim k^{-1}$ in the limit $T \to \infty$.] The finite-exponentiation Lyapunov exponent distribution $Q(h|\lambda)$ can be related to the finite-time Lyapunov exponent distribution $P(h|\tau)$ [11]. It is known that the finite-time Lyapunov exponent distribution $P(h|\tau)$ has an asymptotic form $\sim \exp[-\tau G(h)]$ for a large τ in a chaotic dynamical system, where $G(h)$ has a minimum at $\bar{h}$ with $G(\bar{h}) = G'(\bar{h}) = 0$, $G''(\bar{h}) > 0$ [12]. Utilizing the relationship between h and τ, $h = \lambda/\tau$, a scaling form of $Q(h|\lambda) \sim \exp(-\lambda\frac{G(h)}{h})$ for large λ, can be obtained. (A more detailed discussion of the relationship between $Q(h|\lambda)$ and $P(h|\tau)$ can be found in [10].) For a large λ (corresponding to large k) the integration in (8) can be approximated by

$$\tilde{F}_\phi(k) \sim \frac{1}{k}\int dh\,\exp[-\lambda H(h)] \sim \frac{1}{k^{1+\xi}}\,, \tag{9}$$

where $H(h) = h^{-1}G(h) + (hT)^{-1}$ and $\xi = \min[H(h)] = H(h^*)$, with h^* given by $H'(h^*) = 0$. To verify the theory, $G(h)$ is numerically obtained from a histogram approximation to the finite-time Lyapunov exponent distribution $P(h|\tau)$ which is computed using many wave packets initially spread randomly over space. From the results of this histogram, we obtain $G(h)$ using a cubic polynomial fit to $\tau^{-1}\ln P(h|\tau)$. The assumption of no correlation between the lifetimes of individual wave packets and their finite-time Lyapunov exponents is checked in the following way. We modify Eq. (5) to $d\Omega_j/dt = -2\kappa k_j^2\Omega_j - \Omega_j/T$. Thus (as in the case of a spatially uniform D), all $\Omega_j(\tau_j)$ decay exponentially at

TABLE I. T is the lifetime of the passive scalar, $1 + \xi_m$ is the numerically measured power law exponent, and $1 + \xi_{th}$ is from our theory. The spectral ranges linear fitted to measure the power law exponents are $\log_{10}k = [0.7, 1.7]$, $[0.65, 1.6]$, $[0.65, 3.0]$, and $[0.65, 1.6]$ for the direct numerical simulation, WPM1, WPM2, and WPM3, respectively.

	T	$1 + \xi_m$	$1 + \xi_{th}$
Direct numerical simulation		1.60	1.50
WPM1	7.69	1.60	1.50
WPM2	7.66	1.53	1.50
WPM3	7.69	1.62	1.50

the same rate ($\sim e^{-\tau_j/T}$) independent of the individual dynamics of the wave packets if the diffusivity is negligible. Then, we solve the new set of WPM equations based on the same velocity flow as before. We have obtained essentially identical results for F_ϕ over most of the spectral range. A comparison of results is presented in Table I. WPM1 and WPM2 are from the WPM results presented in Figs. 1 and 2 with diamonds and a solid line, respectively. WPM2 is the same case as WPM1 except that a smaller diffusivity ($\kappa = 2.5 \times 10^{-9}$) is used to verify our theory which is partially obstructed by relatively high diffusivity in WPM1. WPM3 is the case designed to check the independence between the individual lifetime of the wave packet and the finite-exponentiation Lyapunov exponent as described earlier.

The reason why the decrease of the power spectrum is steeper with finite lifetime is that, as scalar variance is transported to larger k, it is also being steadily removed. This picture can be used to obtain a simple, but rough, mean-field-type estimate of ξ, as follows [13]. We neglect fluctuations of the finite-time Lyapunov exponent, and take the stretching to be constant in time at $\bar{h}$ for all orbits, where $\bar{h}$ is the usual (infinite-time) Lyapunov exponent. Thus, in place of (4) we have $dk/dt = \bar{h}k$. Balancing the flux of scalar variance in k space with the loss due to finite lifetime, we have, $\partial/\partial(\bar{h}kF_\phi) = -F_\phi/T$. This gives $\xi_{MF} = (\bar{h}T)^{-1}$. However, since $\xi = \min[H(h)]$ and $H(\bar{h}) = (\bar{h}T)^{-1} = \xi_{MF}$, we see that, in general, ξ is less than $(\bar{h}T)^{-1}$. For example, for the case of Fig. 2, we have $\xi_{MF} = (\bar{h}T)^{-1} = 0.65$, while $\xi = \min[H(h)] = 0.50$. In another case, with smaller T, we have found still larger discrepancy between ξ_{MF} and $\xi = \min[H(h)]$ (with the later agreeing well with the full numerical simulation).

In conclusion, we have obtained a new power law $F(k) \sim k^{-(1+\xi)}$ for the power spectrum of a finite lifetime passive scalar field convected by a chaotic flow [14]. The power law exponent, $-(1 + \xi)$ (different from the Batchelor's k^{-1} law), is found to be determined by the dynamical property of the chaotic flow and the lifetime of the passive scalar.

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