Stability of magnetohydrodynamic Dean Flow as applied to centrifugally confined plasmas

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Dean Flow is the azimuthal flow of fluid between static concentric cylinders. In a magnetized plasma, there may also be radial stratification of the pressure. The ideal magnetohydrodynamic stability of such a flow in the presence of a strong axial magnetic field and an added radial gravitational force is examined. It is shown that both the Kelvin–Helmholtz instability and pressure-gradient-driven interchanges can be stabilized if the flow is driven by a unidirectional external force and if the plasma annulus is sufficiently thin (large aspect ratio). These results find application in schemes using centrifugal confinement of plasma for fusion. © *1999 American Institute of Physics.* [S1070-664X(99)01110-6]

I. MOTIVATION

An idea revived recently is to use centrifugal forces of rotating plasma to augment magnetic confinement of thermonuclear fusion plasmas.[1,2] One possible configuration that employs centrifugal confinement is shown in Fig. 1. The plasma rotates toroidally at supersonic speeds, thus localizing the plasma, from centrifugal forces, to the straight section on the outboard side. Such a configuration is expected to have at most a weak toroidal magnetic field — the poloidal field dominates.

The central question for the success of this scheme is the ideal magnetohydrodynamic (MHD) stability of the rapidly rotating configuration. To this end, there are several considerations that enter. We discuss these below:

(a) The predominant MHD instability is the flute, interchange mode — there is no kink mode in the system since there are no substantial parallel plasma currents. The Kelvin–Helmholtz instability is also a concern.

(b) The free energy for the interchange mode is the pressure gradient. This energy is released by the ''gravitational acceleration'' in the system, in particular the radially outward centrifugal acceleration, $r\Omega^2$, as well as thermal acceleration from the curvature of the field lines[3] as they go from the straight section to the insulators (see Fig. 1). (Ω is the angular frequency.) Because the straight section does not contribute to the curvature, the magnitude of the averaged curvature acceleration is of order $(c_s^2/a)(a/L)=c_s^2/L$, where c_s is the sound speed $\sim$ ion thermal speed. Thus, the combined effective gravity can be written $g_{\mathrm{eff}}\rightarrow r\Omega^2+c_s^2/L$.

(c) While the plasma rotation is destabilizing on account of $r\Omega^2$, the same rotation tends to stabilize the flute mode if there is shear in the angular frequency,[4,5] $d\Omega/dr$. To be sure, this possibility constitutes an added motivation for the idea of centrifugal confinement.[2] The key question remains, though, as to which of the competing effects, centrifugal acceleration versus rotation shear, prevails.

(d) The Kelvin–Helmholtz (KH) instability also has to be considered. The free energy of the KH is the spatial variation in the angular frequency, especially the second derivative, $d^2\Omega/dr^2$. This instability is potentially less serious than the interchange since, as is known from ordinary fluids,[6,7] the stability criterion is profile-dependent and, for reasonable profiles, the mode may be only weakly growing, that is unstable from only nonideal effects such as viscosity.

(e) Finally, one important aspect of the MHD stability of this situation, in contrast to ordinary fluids, is that the strong magnetic field completely stabilizes wavenumbers parallel to the field. Thus, while the fastest growing modes in fluids are azimuthally symmetric rolls in the poloidal plane, in an MHD plasma the only possible instabilities are flutes with $\vec{B}\cdot\vec{\nabla}\approx 0$. This feature makes MHD plasmas more stable than unmagnetized fluids: in particular, conclusions cannot readily be drawn from ordinary fluid stability of, for instance, Dean Flow[7] (described further below).

In this paper, we investigate in greater detail the above seemingly conflicting issues. To be sure, analytical and numerical calculations done to date, using simplified models, suggest that velocity shear is a dominant influence and could significantly suppress the interchange modes.[2,5,8,9] These calculations also suggest that the KH is not a concern given the velocity profiles expected. A complete assessment of stability, for the geometry of Fig. 1 for example, will have to involve a three-dimensional (3D) MHD numerical simulation. Work is in progress along these lines. For the present paper, we simplify the system somewhat and present an analytical study that shows fairly convincingly that a large aspect ratio system could be stabilized.

Our model problem is described in the next section. The stability analysis is given in Sec. III. We summarize our findings in Sec. IV and apply them to centrifugally confined plasmas (CCPs) in Sec. V.

II. THE DEAN FLOW MODEL

The model we use to assess the stability of the system of Fig. 1 is the flow system known in fluid mechanics as Dean Flow.[7] This model is depicted in Fig. 2. Essentially, Fig. 2 is the magnetic field of Fig. 1 straightened out; however, we have added a radially outward external gravitational force to

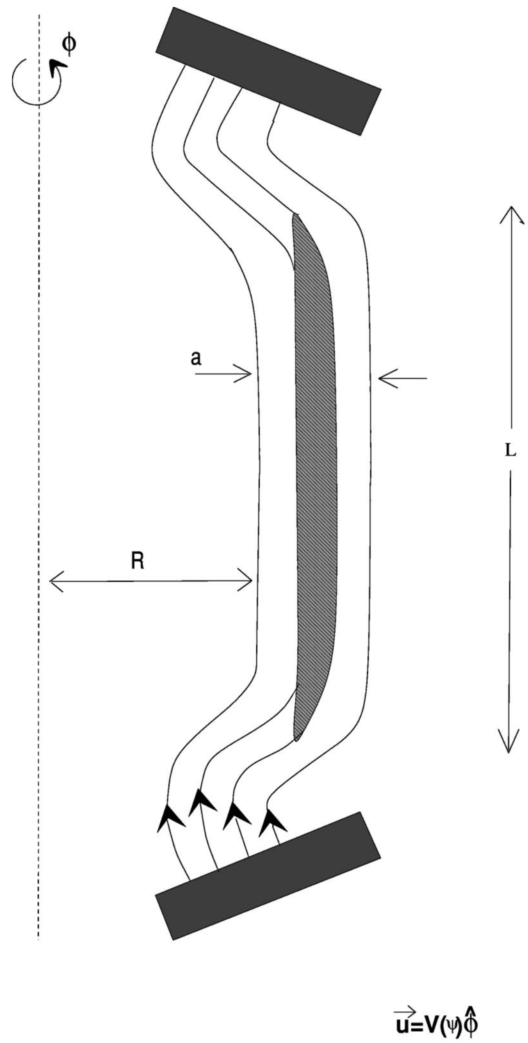

FIG. 1. A possible magnetic configuration for a centrifugally confined plasma.

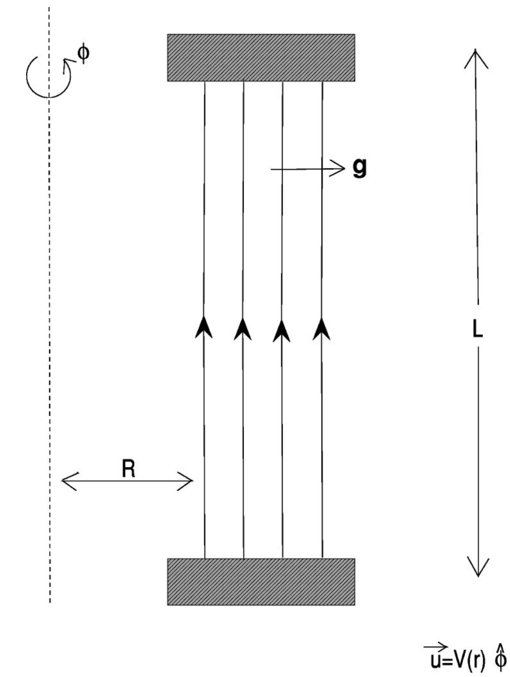

FIG. 2. A model for the configuration of Fig. 1 with an effective gravity, $\vec{g}$, modeling the field line curvature.

model the curvature in the field of Fig. 2. As discussed earlier in part (b) of Sec. I, the magnitude of this artificial gravity, g, is $O(c_s^2/L)$. Such a g force is not normally included in Dean Flow as conventionally defined. The key feature of Dean Flow that distinguishes it from Couette Flow is that the equilibrium azimuthal flow, $\vec{u}=\hat{\theta}V(r)$, is zero at the boundaries[7] (i.e., the boundaries are static and the flow is forced within the fluid). Such a setup is relevant to centrifugally confined plasma schemes. In our model, we will assume that the magnetic field is very strong: more precisely, we will adopt the ordering

$$c_s \sim V \ll V_A,$$

where V_A is the Alfvén speed. This is realistic for CCP plasmas and analytically convenient since it renders the system two-dimensional $(\partial/\partial z=0)$ and incompressible, and allows the use of the MHD reduced equations.

Given these assumptions, the governing equations are the well-known reduced equations of MHD.[10] For the variables n, the mass density, p, the pressure, and ϕ, the flow streamfunction, the equations are given by

$$\frac{\partial n}{\partial t}+\vec{u}\cdot\vec{\nabla} n=0, \tag{1}$$

$$\vec{\nabla}\cdot\left[n\frac{\partial}{\partial t}\vec{\nabla}\phi\right]+n\vec{u}\cdot\vec{\nabla}\nabla^2\phi+\vec{\nabla}n\vec{u}:\vec{\nabla}\vec{\nabla}\phi=T^{-1}\vec{g}\times\vec{z}\cdot\vec{\nabla}p, \tag{2}$$

$$\frac{\partial p}{\partial t}+\vec{u}\cdot\vec{\nabla} p=0, \tag{3}$$

where

$$\vec{u}\equiv\hat{z}\times\vec{\nabla}\phi, \quad p\equiv nT. \tag{4}$$

We have assumed that $\partial_z=0$, in which case the field is completely specified by $\vec{B}\simeq B_0\hat{z}$, $B_0=\text{const}$. We have also assumed that $\vec{g}$ acts only on the pressure, p. In ordinary fluids, gravitational forces act on the density. In a magnetized plasma, it is well-known that the ''effective gravity'' from magnetic curvature terms acts on the pressure.[3,11]

From the above, the equilibrium of Fig. 2 is given by

$$\vec{B}\simeq B_0\hat{z}, \tag{5}$$

$$n=n(r), \quad p=p(r), \tag{6}$$

$$\vec{u}=\hat{\theta}V(r), \quad V(r)\equiv d\phi/dr. \tag{7}$$

We will also use the angular frequency $\Omega(r)$, defined according to $V\equiv r\Omega$.

III. STABILITY ANALYSIS

We now linearize (1)–(3) about the above equilibrium. We assume perturbations of the form $n\rightarrow n(r)+\tilde{n}(r)\exp[im\theta-i\omega t]$. The resulting equations for $\tilde{n}$, $\tilde{\phi}$, and $\tilde{p}$, after some algebra, are

$$\bar{\omega}\tilde{n}+(m/r)n'\tilde{\phi}=0, \tag{8}$$

$$\bar{\omega}\vec{\nabla}\cdot[n\vec{\nabla}\tilde{\phi}]+(m/r)(nf)'\tilde{\phi}=(m/r)[r\Omega^2\tilde{n}+g\tilde{p}/T], \tag{9}$$

$$\bar{\omega}\tilde{p}+(m/r)p'\tilde{\phi}=0, \tag{10}$$

where

$$f(r)\equiv(r^2\Omega)'/r, \tag{11}$$

$$\bar{\omega}\equiv\omega-m\Omega, \tag{12}$$

and primes denote differentiation with respect to r. The system has to be solved subject to the boundary conditions $\tilde{\phi}(R)=0$, $\tilde{\phi}(R+a)=0$. Here, R is the radius of the inner cylinder and a is the width of the annulus.

We begin our stability analysis by first considering some special cases.

A. Rayleigh's theorem

In the limit $n'=0$ and $p'=0$, $\tilde{n}$ and $\tilde{p}$ vanish and the system reduces to the familiar Kelvin–Helmholtz eigenvalue equation,[6,7] viz.

$$\bar{\omega}\nabla^2\tilde{\phi}+(m/r)f'\tilde{\phi}=0. \tag{13}$$

An important theorem, akin to Rayleigh's inflexion-point theorem for plane parallel flow in fluids,[6,7] is readily obtained from (13) by constructing a quadratic form. Dividing by $\bar{\omega}$ and operating on (13) with $\int_R^{R+a}dr\, r\tilde{\phi}^*$, integrating by parts, and using the homogeneous boundary conditions, we obtain

$$\langle|\vec{\nabla}\tilde{\phi}|^2\rangle=\left\langle\frac{mf'|\tilde{\phi}|^2}{r\bar{\omega}}\right\rangle, \tag{14}$$

where $\langle h\rangle\equiv\int dr\, h$. The imaginary part of (14) is

$$0=\gamma\left\langle\frac{mf'|\tilde{\phi}|^2}{r|\bar{\omega}|^2}\right\rangle, \tag{15}$$

where we have used $\omega=\omega_r+i\gamma$. It follows that if $\gamma\neq0$, f' must vanish somewhere in the domain. Conversely, the system is marginally stable if $f'\neq0$, i.e.,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2\Omega)\right]\neq0=>\text{marginal stability.} \tag{16}$$

Thus, the system is ideally stable if $f'\neq0$, although it could be unstable to nonideal modes mediated by viscosity.

B. Simple interchange mode

Now suppose that n' and p' are both nonzero but that the angular frequency is constant. The only effect of Ω, then, should be to provide a centrifugal acceleration and the instability should resemble a Rayleigh–Taylor-type interchange mode. Indeed, if $\Omega=$const., we obtain the system

$$\bar{\omega}\vec{\nabla}\cdot[n\vec{\nabla}\tilde{\phi}]+2(m/r)n'\Omega\tilde{\phi}=(m/r)[r\Omega^2\tilde{n}+g\tilde{p}/T], \tag{17}$$

$$\bar{\omega}\tilde{n}+(m/r)n'\tilde{\phi}=0, \tag{18}$$

$$\bar{\omega}\tilde{p}+(m/r)p'\tilde{\phi}=0. \tag{19}$$

If we let $m\to\infty$ in (17) and (19), the ansatz $d/dr \ll(m/r)$ is valid and the system reduces to

$$\bar{\omega}^2\nabla^2\tilde{\phi}\to-(m/r)^2[r\Omega^2n'/n+gp'/p]\tilde{\phi}, \tag{20}$$

from which we obtain the well-known ''local dispersion relation'' for the usual simple interchange[11]

$$\bar{\omega}^2=r\Omega^2(n'/n)+g(p'/p). \tag{21}$$

In what follows, we will examine the stability of the full system (8)–(10) by methods similar to the quadratic form and the local dispersion shown above. For this paper, we have been unable to assess stability in the completely general case. Consequently, we will only present a stability analysis for the large aspect ratio limit $(R/a)\gg1$. In this limit, one general observation may be made from (8)–(10). Since $R\gg a$, we may also assume $d/dr\gg1/r$ in (8)–(10). In that case, if we assume further that $m\sim O(1)$ and that $\omega\sim m\Omega$, it is readily seen that the three (grouped) terms in (9) scale as

$$1:1:(a/R). \tag{22}$$

Based on (22), we will treat the right-hand-side (RHS) of (9) as a perturbation in the limit $R/a\gg1$.

Before proceeding, we note that the generalization of the quadratic form (14) can be readily obtained. We find

$$\langle n|\vec{\nabla}\phi|^2\rangle=\left\langle\frac{m|\tilde{\phi}|^2(nf)'\bar{\omega}^*}{r|\bar{\omega}|^2}\right\rangle+\left\langle\left[r\Omega^2\frac{|\tilde{n}|^2}{n'}+\frac{g|\tilde{p}|^2}{Tp'}\right]\frac{\bar{\omega}^*}{\bar{\omega}}\right\rangle, \tag{23}$$

where the second term on the RHS is a small term for $R>>a$. Note that the n' and p' terms in the denominators of this term are not an issue, for as $n'\to0$, $\widetilde{n\to}0$ as n' [see Eq. (8)]; likewise for $p'\to0$, thus there is no singularity. The imaginary part of the quadratic form becomes

$$0=-\gamma\left\langle\frac{m|\tilde{\phi}|^2(nf)'}{r|\bar{\omega}|^2}\right\rangle+\left\langle\left[r\Omega^2\frac{|\tilde{n}|^2}{n'}+\frac{g|\tilde{p}|^2}{Tp'}\right]\mathrm{Im}\left(\frac{\bar{\omega}^*}{\bar{\omega}}\right)\right\rangle, \tag{24}$$

where we can substitute

$$\mathrm{Im}\left(\frac{\bar{\omega}^*}{\bar{\omega}}\right)=-\frac{2\gamma\omega_r}{|\bar{\omega}|^2}.$$

C. Long wavelength modes for $R/a\gg1$

For the long wavelength ordering $m\sim O(1)$, $d/dr\sim1/a$ (both for equilibrium as well as perturbed quantities), and if $R\gg a$, we may use the quadratic form (24) to assess stability. To do this, we write (24) as

$$\gamma[A+B]=0, \tag{25}$$

where $B/A \sim O(a/R)$. Now if $(nf)' \neq 0$, then A is nonzero and B may be neglected for $a/R \ll 1$. It follows that $\gamma = 0$ to all orders. Thus, we obtain the modified version of Rayleigh's theorem:

$$\frac{d}{dr}\left[\frac{n}{r}\frac{d}{dr}(r^2\Omega)\right] \neq 0 => \text{marginal stability.} \tag{26}$$

D. Short wavelength modes, $m \gg 1$, for $R/a \gg 1$

We now consider short wavelength modes, $m \gg 1$, for a large aspect ratio system. For $m \gg 1$, we may make the ''local'' ansatz $|d\tilde{\phi}/dr| \ll (m/r)|\tilde{\phi}|$, but d/dr on equilibrium quantities $\sim 1/a$. In that case (8)–(10) simplify to

$$\frac{d}{dt}\nabla^2\tilde{\phi} = -\Omega^2\frac{\partial}{\partial\theta}\left(\frac{\tilde{n}}{n}\right) - \frac{g}{r}\frac{\partial}{\partial\theta}\left(\frac{\tilde{p}}{p}\right), \tag{27}$$

$$\frac{d}{dt}\left(\frac{\tilde{n}}{\tilde{n}}\right) = \left(\frac{n'}{nr}\right)\frac{\partial\tilde{\phi}}{\partial\theta}, \tag{28}$$

$$\frac{d}{dt}\left(\frac{\tilde{p}}{p}\right) \simeq \left(\frac{p'}{pr}\right)\frac{\partial\tilde{\phi}}{\partial\theta}, \tag{29}$$

where $d/dt \equiv \partial/\partial t + \Omega\,\partial/\partial\theta$ and we have reverted to partial derivatives. We will let $r \to R + x$, thus, $\nabla^2 \to \partial_x^2 + R^{-2}\partial_\theta^2$.

As in a previous calculation,[5] we now make the following transformations

$$X = x, \tag{30}$$

$$\xi = \theta - \Omega t, \tag{31}$$

$$\tau = t. \tag{32}$$

In the new coordinates, we have

$$\frac{\partial}{\partial\tau}\nabla^2\tilde{\phi} = -\frac{\Omega^2}{R}\frac{\partial}{\partial\xi}\frac{\tilde{n}}{n} - \frac{g}{R}\frac{\partial}{\partial\xi}\frac{\tilde{p}}{p}, \tag{33}$$

$$\frac{\partial}{\partial\tau}\frac{\tilde{n}}{n} = \frac{n'}{n}\frac{1}{R}\frac{\partial\tilde{\phi}}{\partial\xi}, \tag{34}$$

$$\frac{\partial}{\partial\tau}\frac{\tilde{p}}{p} = \frac{p'}{p}\frac{1}{R}\frac{\partial\tilde{\phi}}{\partial\xi}, \tag{35}$$

where

$$\nabla^2 \equiv \left[\left(\frac{\partial}{\partial X} - \Omega'\tau\frac{\partial}{\partial\xi}\right)^2 + \frac{1}{R^2}\frac{\partial^2}{\partial\xi^2}\right]. \tag{36}$$

In these coordinates, X is ignorable. The fastest growth rates are obtained for $\partial/\partial X = 0$. We thus look for modes with $\partial/\partial X = 0$. $(\partial/\partial\xi)$ is also ignorable: we let $\partial/\partial\xi \to im$. We then obtain

$$\frac{\partial}{\partial\tau}[1 + (R\Omega'\tau)^2]\frac{im\tilde{\phi}}{R} = -R\Omega^2\frac{\tilde{n}}{n} - \frac{g}{R}\frac{\tilde{p}}{p}, \tag{37}$$

$$\frac{\partial}{\partial\tau}\frac{\tilde{n}}{n} = \frac{n'}{n}\frac{im}{R}\tilde{\phi}, \tag{38}$$

$$\frac{\partial}{\partial\tau}\frac{\tilde{p}}{p} = \frac{p'}{p}\frac{im}{R}\tilde{\phi}. \tag{39}$$

Eliminating $\tilde{n}$ and $\tilde{p}$, we get for $\tilde{\phi}$

$$\frac{\partial^2}{\partial\tau^2}[1 + (R\Omega'\tau)^2]\tilde{\phi} = -\left[R\Omega^2\frac{n'}{n} + g\frac{p'}{p}\right]\tilde{\phi}. \tag{40}$$

For $R\Omega^2 n'/n + gp'/p > 0$, the system is stable. For this quantity negative, the system is unstable. As discussed in detail in Ref. 5, on short time scales, $R\Omega'\tau \ll 1$, the mode grows exponentially as $\exp[\gamma_g t]$ with

$$\gamma_g = [-R\Omega^2 n'/n - gp'/p]^{1/2}. \tag{41}$$

On longer time scales, $R\Omega'\tau \gg 1$, the mode still grows but only algebraically, as τ^α. If viscosity and resistivity are included, the algebraic growth is efficiently phase mixed by the velocity shear, resulting in stabilization. A ''stability criterion'' can be written if $R\Omega' > \gamma_g$. This is given by[5]

$$R^2\Omega'^2 > \gamma_g^2 \ln[R_\mu] \tag{42}$$

where $R_\mu \equiv \gamma_g r^2/m^2(\mu\eta)^{1/2}$ is a Reynolds number.

Between (41) and (42), we may now assess the stability of a system such as Fig. 1 if we let $g \to c_s^2/L$, for $L \gg a$. Schematically, if $V' = (R\Omega)' \sim R\Omega/a$, we have the schematic stability criterion

$$\frac{R}{a} > \left[\frac{a}{L_n} + \frac{1}{M_s^2}\frac{R}{L}\frac{a}{L_p}\right]\ln R_\mu, \tag{43}$$

where L_n is the density scale size, L_p is the pressure scale size, L/a is the elongation, and $M_s \equiv V/c_s \sim R\Omega/c_s$ is the Mach number. For $L_n \sim a$ and $L_p \sim a$, the centrifugal force is the dominant destabilizing mechanism for $M_s^2 > (R/L)$ while the curvature gravitational force dominates for $M_s^2 < R/L$. We will discuss all these in reference to CCPs in Sec. V.

As a final remark, we note that the stability criterion (42), taken from the work in Ref. 4, is based on a very conservative interpretation of the mode growth. Only a numerical simulation can quantify this condition in more realistic terms.

IV. SUMMARY OF FINDINGS

We have investigated the stability of Dean Flow in MHD in the limit of large aspect ratio. We can summarize our findings as follows:

(i) long wavelength modes, with $m \sim O(1)$ to $m \sim O(R/a)$, are primarily KH-type modes. These can be made marginally stable, to ideal perturbations, if the Rayleigh criterion

$$\frac{d}{dr}\left[\frac{n}{r}\frac{d}{dr}(r^2\Omega)\right] \neq 0, \tag{44}$$

is satisfied throughout the plasma. As we will show below, this condition is achieved in Dean Flow and, therefore, for CCPs. It is possible that nonideal instability can occur, mediated by viscosity.[6,7] These instabilities are expected to grow at a rate of order Ω/R_μ^ν, where R_μ is the viscous Reynolds number and ν is a fractional power. This is sufficiently slow, that a weak toroidal magnetic field, $B_T \ll B_p$, could be used to stabilize such instability since a toroidal magnetic field adds magnetic shear to the system which provides stabilizing Alfvénic forces.[11] The magnetic shear results in a stabilizing frequency that scales as $(B_T/B_p)V_A/L$; since the viscous KH will have a growth rate of order Ω/R_μ^ν, it follows that a very weak B_T will suffice since $R_\mu \gg 1$.

(ii) Short wavelength modes, i.e., modes with $m \gg R/a$, are primarily flute interchange modes. These can be stabilized by phase mixing from velocity shear. A conservative stability criterion is given by (42) or (43).

It is useful here to elaborate on the short wavelength stability of the system. First, we recall that the entire calculation of this paper has been based on the MHD ordering, by which we mean we are considering situations where the ion Larmor radius is very small, i.e., $\rho_i \ll a$. Thus, by ''short wavelengths'' in (ii) above, we mean wavelengths short compared to a but much longer than ρ_i. A second assumption of MHD ordering is that the frequencies under consideration must be very low compared to the ion cyclotron frequency and of order the sonic and Alfvénic frequencies. The upper bound in the frequencies is self-consistent with $\rho_i \ll a$. It is clear that stability of a magnetic confinement system on the sonic and Alfvénic timescales must be established (lack thereof is tantamount to no equilibrium) — this we have done in this paper. In addition, dissipative effects are stabilizing for these ideal modes.

The MHD ordering, however, precludes consideration of mode stability at frequencies lower than sonic/Alfvénic frequencies. In particular, the next natural frequency downscale in a magnetized plasma is the drift frequency, ω_*, which is greater than or of order ρ_i/a times the sonic frequency. Our paper has not dealt with this frequency scale. Drift modes in a plasma can be destabilized because of dissipative effects (in contrast to the stabilization tendency of dissipative effects on the MHD modes discussed above). These modes, however, have lower growth rates ($\sim\omega_*$), peaking at shorter wavelengths ($\sim\rho_i$), and do not lead to a catastrophic loss of equilibrium — rather, they create small scale turbulence. The latter turbulence leads to nontrivial heat loss from fusion plasmas and has to be accounted for in transport considerations. This is outside the scope of our paper. However, it is worth noting here that the drift modes may be strongly stabilized by the velocity shear we are considering. In brief, the velocity shear frequency, V', we consider here is larger than sonic frequencies and so, by definition, it is much larger than drift frequencies: if the interchanges can be stabilized, the drift mode is very likely to be stabilized at the large velocity shears.

V. APPLICATION TO CENTRIFUGALLY CONFINED PLASMAS

As has been observed elsewhere,[1,2] the centrifugal forces from a supersonic rotating plasma can be used to effect confinement along the magnetic field. This is one of the underlying motivations to use CCPs for thermonuclear fusion. Another motivation is that the large velocity shear would suppress instabilities, possibly even highly potent ideal MHD instabilities such as flute interchanges. In this paper we have assessed the stability of CCPs with respect to ideal MHD instabilities. Based on the Dean Flow model, we conclude that long wavelength modes are stable for large aspect ratio if the Rayleigh criterion is satisfied. The latter criterion is, in fact, readily satisfied, for this is just the condition one expects if the toroidal flow is driven by an external force in the plasma, as follows. The steady state equation for flow as a balance between viscous stresses and an external force, $\vec{F}$, is given by[12]

$$0=\vec{\nabla}\cdot\{n\mu[(\vec{\nabla}\vec{u})+(\vec{\nabla}\vec{u})^T]\}+\vec{F}, \tag{45}$$

where μ is the viscosity. If we let $\vec{u}=\hat{\theta}r\Omega$ in this equation, and let $\vec{F}=\hat{\theta}F_\theta$, we obtain

$$\frac{d}{dr}\left[\frac{n\mu}{r}\frac{d}{dr}(r^2\Omega)\right]=-F_\theta. \tag{46}$$

If $\mu=$ constant and $F_\theta\neq 0$, clearly Rayleigh's criterion is satisfied everywhere in the plasma.

Short wavelength mode stability is given by (43). Now for CCPs, we need large Mach numbers,[2] $M_s \gtrsim 3.5$. Thus, it is reasonable to assume that $M_s^2 > R/L$, in which case the centrifugal term in (43) dominates if $n' \sim n/a$. The stability criterion then reduces to

$$\left(\frac{R}{a}\right) > \left(\frac{a}{L_n}\right)\ln R_\mu. \tag{47}$$

This condition may not be easy to implement for high temperature plasma since R_μ will be large. [For typical fusion parameters, $\ln(R_\mu)$ can be as large as 16.] The condition, however, is based on a very conservative interpretation of a simple calculation[5] — a numerical simulation is in order. Note that a large aspect ratio is stabilizing. The reason large aspect ratio is more stabilizing is because the stabilization happens when velocity shear, V', overcomes the interchange growth from centrifugal acceleration: in particular, V' scales as $1/a$, for given V, while the growth rate, γ_g, scales as $1/(aR)^{1/2}$, for given V. How much a toroidal magnetic field aids in stabilizing the above is not known at present.

Now it is possible that condition (47) may not be satisfied, in which case short wavelength interchanges will be precipitated. There is, however, a major difference here than in tokamaks — namely, for CCPs, the above interchange is driven by n', as opposed to p' for tokamaks. Thus, the interchanges would lead to a flat density profile. This would not be a fundamental limitation to achieving fusion energy break even if a temperature gradient can be maintained. The latter may be much easier; if we consider (43) as $n' \to 0$, we find the criterion for stability with respect to T'

$$M_s^2 > \left(\frac{a}{L_T}\right)\frac{\ln R_\mu}{(L/a)}. \tag{48}$$

If $L_T \sim a$, then the required elongation for maintenance of T' is

$$\frac{L}{a} > \left(\frac{\ln R_\mu}{M_s^2}\right). \tag{49}$$

Since $M_s^2 \sim 12$, this is quite reasonable.

In this paper, we have come to some conclusions regarding the stability of CCPs based on our model Dean Flow calculation. Dean Flow has some characteristics that make it a good model for CCPs — toroidal geometry, toroidally outward centrifugal force, shear in angular frequency — and we have added an artificial gravity to mimic the thermal acceleration forces arising from the field line curvature of CCPs. The Dean Flow model has the shortcoming that it does not account for the tapered, centrifugally confined pressure along the field line that is the case for a CCP — in a CCP, the centrifugal forces will confine pressure so that it is localized to the straight portion of the magnetic field shown in Fig. 1. This and possibly other geometric effects need to be considered before a definitive assessment of CCP stability can be made. In addition, given the conservative nature of the ''stability criterion'' used in arriving at (42), it is clear that a numerical simulation of a CCP plasma is needed to resolve the stability question more quantitatively. Work is in progress along these lines.

In summary, CCPs at supersonic speeds may be stable to maintenance of pressure gradients to all ideal MHD modes, for modest elongation and large aspect ratio. A large aspect ratio is generally stabilizing since velocity shear scales as $1/a$ while destabilizing accelerations scale as $1/R$. It is possible that the large aspect ratios required to stabilize all modes (cf. Eq. 47) may not be practical. Even so, the resulting instability will flatten only the density gradient but not the temperature gradient. The latter can be maintained in stable state for modest elongations (cf. Eq. 49). From the viewpoint of fusion, maintaining the temperature gradient is the critical requirement.

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