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LINE-TYING AND THE REDUCED EQUATIONS OF MAGNETOHYDRODYNAMICS

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ABSTRACT

The "reduced equations" of MHD are based on the ordering that variations transverse to the magnetic field are sharper than longitudinal variations. We show that this ordering breaks down near conducting surfaces satisfying "line-tied" boundary conditions. A boundary layer is shown to exist there. Reduced equations are derived for the layer and matched asymptotically to the usual equations away from the surface. It is shown that the boundary layer does not affect the usual procedures to solve for the flux surfaces and the flow streamlines.

Subject headings: line: formation — MHD — Sun: magnetic fields

1. INTRODUCTION

Many solar applications are concerned with the dynamics of large-scale magnetic structures in the corona. Loosely speaking, these structures are nearly uniform along some direction. For example, loop structures appear to be long and thin, so that the observed striations are nearly uniform parallel to the loop, but sharp variations occur perpendicular to the loop. More formally, if the field $\boldsymbol{B}$ is oriented primarily along the $\hat{z}$ direction, then it seems appropriate to order $B_z \gg |\boldsymbol{B}_\perp|$ and $L_\perp \ll L_z$, where L denotes the length scale. This "quasi-symmetry" along $\hat{z}$ allows for a reduced description of the system. By applying the above ordering to the governing ideal MHD equations, a "reduced" set of equations (Strauss 1976) is obtained that greatly simplifies the task of investigating these kinds of structures.

Another aspect of these applications involves the dense plasma in the photosphere under the base of the corona. Since the magnetic structures under consideration thread into this dense plasma, the bottom of the corona can be modeled as a boundary condition for the system. Since the plasma is dense and nearly perfectly conducting, the magnetic field is frozen into it and convected by motions generated deep inside the convection zone. This is usually referred to as "line-tying." As the photospheric plasma moves around, the field "footpoints" move with it, driving the motion in the corona. For general footpoint motions, the line-tied boundary conditions at the corona-photosphere interface are $[\boldsymbol{B} \cdot \hat{\boldsymbol{n}}] = 0$ and $[\boldsymbol{E} \cdot \hat{\boldsymbol{t}}] = 0$, where $\boldsymbol{E}$ is the electric field, $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{t}}$ are unit vectors normal to and tangential to the interface, respectively, and [ ] denotes the jump across the surface (Jackson 1962).

In this paper, we show that the ordering $L_\perp \ll L_z$, or equivalently $\nabla_\perp \gg \partial_z$, is incompatible with line-tied boundary conditions for incompressible footpoint motions. Specifically, this ordering prevents the boundary condition on B_z (here $B_z = \boldsymbol{B} \cdot \hat{\boldsymbol{n}}$) from being satisfied. This implies the presence of a boundary layer at the photosphere-corona interface where we show that the $\nabla_\perp \gg \partial_z$ ordering is violated. In the layer, we show that the ordering $\partial_z \sim \nabla_\perp$ is the correct ordering.

The reduced equations of MHD are widely used for solar coronal applications. Generally, however, the possible presence of a boundary layer is not acknowledged or resolved when these equations are applied. Thus, the usual reduced equations are solved and subjected to the "line-tied" boundary conditions at the base of the corona. If, however, there were a boundary layer, the MHD variables in the layer would, in general, be governed by a new and different set of equations, the "layer equations." As such, the procedure for solar coronal problems, even with $L_\perp \ll L_z$ in the corona, would be to (Bender & Orszag 1978) (1) solve the usual reduced equations for the corona, (2) solve separately the layer equations subject to line-tied boundary conditions, and (3) perform an asymptotic matching in the overlap regions.

In principle, the imposition of line-tying via a boundary layer as above could result in different physics than if the boundary layer were ignored. This issue takes on added significance due to the fact that the reduced equations are involved in one of the more compelling controversies in solar physics. Specifically, Parker's (1994) proposal for coronal heating is that footpoint motions, on scales $L_\perp \ll L_z$, lead to steadily evolving neighboring equilibria, which may not necessarily exist free of discontinuities. In particular, sheet discontinuities in the magnetic field arise—"sheet currents"—and these dissipate rapidly to heat the corona. van Ballegooijen (1985), on the other hand, has presented a calculation to show that smooth neighboring equilibria always exist. His calculation assumes $L_\perp \ll L_z$ as well as $B_z \gg |\boldsymbol{B}_\perp|$; in fact, his calculation constitutes an independent derivation of the MHD reduced equations (Strauss 1976). The calculation represents a disagreement with Parker's calculation and provides a basis for the controversy. As such, the possible presence of a boundary layer becomes more significant in light of this controversy insofar as whether it affects the calculation of van Ballegooijen (1985).

It is important to clarify that the aforementioned controversy does not necessarily hinge upon the usage of the reduced equations. To be sure, Parker's (1994) calculations do not necessarily rely on a reduced ordering. In addition, another counterargument to Parker's scenario does not rely on the reduced ordering (Antiochos 1987). Thus, our motivation is simply to assess whether the presence of a boundary layer in any way mitigates the proof of van Ballegooijen, with the awareness that this would address only one significant aspect of this topic.

To return to our main discussion, we present in this paper a detailed calculation to assess the affect of a boundary layer on conclusions drawn from the usual reduced equations. We rederive the reduced equations in the manner of Strauss (1976) and van Ballegooijen (1985), demonstrate the inevitability of a boundary layer, derive new equations for MHD variables in the layer, and perform an asymptotic matching. We conclude that a boundary layer exists and that B_z should be handled with care, but that the "usual reduced equations" for $B_\perp$ and the flow streamlines are uniformly valid throughout the corona and the boundary layer.

The paper is organized as follows: The standard reduced equations are derived for a simple quasistatic system in § 2. Here it is shown exactly where the ordering breaks down. In § 3 the boundary layer equations are derived. Here it is shown that the solution satisfies the boundary conditions and also matches to the interior, "coronal" region. We conclude in § 4. In the Appendix, we give a complete analytic solution for small perturbations demonstrating the inadequacy of the reduced equations.

2. THE REDUCED EQUATIONS

Consider a standard system often used to describe the solar corona (Parker 1994). A magnetic field, nearly uniform along the z-axis, is embedded in an ideal, uniform plasma that exists between two perfectly conducting plates perpendicular to the z-axis at $z = 0$, L_z, respectively. The plates are infinite in extent, and the plasma is initially in equilibrium. Let the plates be subject to quasistatic, incompressible deformations and/or displacements in the x-y plane. Our goal is to "reduce" the ideal MHD equations to a simpler set that adequately describes this system.

For a quasistatic, low-β ideal MHD system, the relevant equations are

$$\boldsymbol{J} \times \boldsymbol{B} = 0 , \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) , \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0 . \tag{3}$$

Here $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{u}$ is the plasma flow, and $\boldsymbol{J} \equiv (c/4\pi)\nabla \times \boldsymbol{B}$ is the current. We are concerned only with short-scale transverse displacements, i.e., $L_\perp \ll L_z$, where $L_\perp$ is the scale length of variation in the x-y plane. For simplicity, we assume the plasma β, the ratio of thermal to magnetic energy, is vanishingly small. We also assume this for the ratio of plasma flow speeds to the Alfvén speed; this is essentially equivalent to defining the plasma as quasistatic.

We now introduce the following ordering (Strauss 1976; van Ballegooijen 1985) in terms of smallness parameter ϵ:

$$\frac{L_\perp}{L_z} \sim \frac{|\boldsymbol{B}_\perp|}{|B_z|} \sim \frac{|u_z|}{|\boldsymbol{u}_\perp|} \sim \epsilon , \tag{4}$$

$$|\boldsymbol{u}_\perp| \sim L_\perp \partial_t . \tag{5}$$

From equation (4), it is reasonable to assume that $(\partial_z/|\nabla_\perp|) \sim \epsilon$. Thus we expect that variation in the x-y plane will have a much sharper scale than in the z-direction, since the plate separation distance L_z is large compared to the displacement of the plates. Although this *Ansatz* is compelling, it is not strictly valid and will in fact lead to failure of the equations at the plates.

We begin by deriving reduced equations in accordance with equations (4) and (5). Since this has been done previously (Strauss 1976; van Ballegooijen 1985), we will be brief. To lowest significant order, equations (1), (2), and (3) yield, respectively,

$$B_z = B_0 = \text{const.} , \tag{6}$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp = 0 , \tag{7}$$

$$\nabla_\perp \cdot \boldsymbol{B}_\perp = 0 . \tag{8}$$

From equations (7) and (8), we immediately make the definitions

$$\boldsymbol{u}_\perp \equiv \hat{z} \times \nabla_\perp \varphi , \tag{9}$$

$$\boldsymbol{B}_\perp \equiv \hat{z} \times \nabla_\perp \psi . \tag{10}$$

Using the definitions in equations (9) and (10) in equation (2), taken to the next order, we obtain

$$\frac{\partial \psi}{\partial t} = \boldsymbol{B} \cdot \nabla \varphi , \tag{11}$$

which is the perpendicular component of Faraday's Law. Here $\boldsymbol{B} \cdot \nabla \equiv B_0 \partial_z + \boldsymbol{B}_\perp \cdot \nabla_\perp$. We also obtain

$$B_{z_1} = \text{const} , \tag{12}$$

where $B_z = B_0 + B_{z_1} + B_{z_2} + \cdots$, with each term smaller than the previous by a factor ϵ. Since B_0 and B_{z_1} are both constants, we can simply absorb B_{z_1} into B_0 and set B_{z_1} to zero.

To second order in ϵ, we obtain two equations by operating on the force balance equation (1) with $\nabla_\perp \times$ and $\nabla_\perp \cdot$, respectively. Taking the curl yields

$$\boldsymbol{B} \cdot \nabla(\nabla_\perp^2 \psi) = 0 , \tag{13}$$

which implies that currents do not vary along the magnetic field. Taking the divergence gives

$$B_0 \nabla_\perp^2 B_{z_2} = -\nabla_\perp \cdot (\nabla_\perp \psi \nabla_\perp^2 \psi) ; \tag{14}$$

as an equation for B_{z_2}, where $\nabla_\perp^2 \equiv \partial_x^2 + \partial_y^2$.

At this juncture, it is useful to recall that we are assuming that the system is of low plasma β. From equation (14), we infer that $\beta \equiv p/B^2$ must be smaller than ϵ^2. The low-β assumption has been made for simplicity: it can be shown that including β does not alter the conclusions of this paper.

Our system is now governed by equations (11), (13), and (14) for the variables φ, ψ, and B_{z_2}. The procedure to solve for these variables is as follows (van Ballegooijen 1985): One first integrates equation (13) along a field line to obtain $\nabla_\perp^2 \psi$ in terms of a free function J. From this, ψ can be calculated, in principle. One then inserts ψ into equation (11), which, in turn, is integrated along the field line to obtain φ. The value of φ is known on both the plates, i.e., at both end points of the latter integration. Since equation (11) is only a first-order ordinary differential equation along the field, the two boundary conditions constitute a constraint on the free function J, which, in principle, is thus determined. Then ψ can be found. In this way, ψ and φ are solved for from equations (11) and (13).

A problem arises, however, in solving for B_{z_2}. As stated earlier, B_z must be continuous at the plates. If the initial $\boldsymbol{B}$ field inside the plates is uniform and in the z-direction, then, for incompressible transverse deformations of the "plate," the $\boldsymbol{B}$ field inside is simply convected and does not change. Thus, B_{z_2} must be zero at the plates. As can be seen from equation (14), the z-dependence of B_{z_2} depends directly on the z-dependence of ψ. As such, there is no guarantee, in general, that B_{z_2} will vanish at the plates. We demonstrate this fact by a complete calculation in the linearized limit in the Appendix. The upshot is that while B_z can be solved for from equation (14), it does not satisfy the boundary conditions. As is well known from asymptotic analysis (Bender & Orszag 1978), this failure indicates the presence of boundary layers. We are therefore forced to reorder the equations in the layers near the plates. Note that these layers develop even if the plate motions are incompressible; we will discuss compressible motions (Zweibel & Boozer 1985; Zweibel & Li 1987) later.

3. BOUNDARY LAYER EQUATIONS

We have shown that equations (6)–(14) do not permit a solution to the magnetic field that satisfies line-tied boundary conditions, even for small perturbations about a straight $B_z = B_0$ field (Appendix). This indicates the presence of boundary layers at the plates. In this section, we derive the equations that φ, ψ, and B_{z_2} satisfy in the boundary layers. As discussed in the Introduction, such a derivation is imperative. For if it was found that φ and ψ satisfy different equations in the layers, a distinct possibility, that would imply that implications drawn for solar coronal problems from equations (11) and (13) need reconsideration. In particular, as we will show below, the ordering $\nabla_\perp \gg \partial_z$, used in deriving equations (11) and (13), fails in general in the boundary layers. The correct ordering for the layers is

$$\partial_z \sim \nabla_\perp \; . \tag{15}$$

The rest of the ordering scheme (4)–(5) is unaffected. In what follows, to keep the terminology consistent we will use "layer" to mean the boundary layer and "corona" to refer to the interior region $z = (0, L_z)$ away from the layer.

Let us use the revised ordering scheme of equation (15) to rederive equations for φ^*, ψ^*, and B_z^* in the layers, where the superscript * denotes a layer quantity. To lowest order in ϵ, equations (1) and (3) yield

$$B_z^* = \text{const.} = B_0 \; , \tag{16}$$

where the constant must be the same as in equation (6). From equation (2) to lowest significant order, we find $\nabla_\perp \cdot \boldsymbol{u}_\perp^* = 0$ and $\partial_z \boldsymbol{u}_\perp^* = 0$, from which we conclude

$$\boldsymbol{u}_\perp^* \equiv \hat{z} \times \nabla_\perp \varphi^* \; , \tag{17}$$

$$\frac{\partial \varphi^*}{\partial z} = 0 \; . \tag{18}$$

Equation (18) is the layer equation for φ^*. Note that for this system there are two layers: one near the top plate at $z = L_z$ and one near the bottom plate at $z = 0$. Equation (18) must therefore be solved for each layer independently using the relevant plate boundary condition. For simplicity, in what follows we solve only for the layer near $z = 0$; the solution for the other layer near $z = L_z$ is identical. For the layer near $z = 0$ then, equation (18) must be solved subject to the boundary condition that φ^* must equal the plate potential at $z = 0$ and that φ^* must match asymptotically to φ in the corona. The boundary condition gives

$$\varphi^*(\boldsymbol{x}_\perp, t) = \varphi_{\text{plate}}(\boldsymbol{x}_\perp, t) \; , \tag{19}$$

and the matching condition, therefore, is

$$\varphi(\boldsymbol{x}_\perp, z \to 0, t) \to \varphi_{\text{plate}}(\boldsymbol{x}_\perp, t) \; . \tag{20}$$

The significance of equations (19) and (20) is that φ in the "corona" effectively satisfies a boundary condition (eq. [20]), which is just as if there were no boundary layer. We thus conclude that the boundary layer does not affect the solution of φ as obtained with the usual reduced equations.

To continue, from equations (1) and (3) taken to the next order, we have

$$\nabla B_{z\,1}^* = B_0\, \partial_z(\hat{z} B_{z\,1}^* + \boldsymbol{B}_\perp^*) \; , \tag{21}$$

$$\partial_z B_{z\,1}^* + \nabla_\perp \cdot \boldsymbol{B}_\perp^* = 0 \; , \tag{22}$$

from which we conclude the following:

$$\nabla^2 B_{z\,1}^* = 0 \; . \tag{23}$$

Since B_{z_1} is zero at the plates and zero in the corona, $B_{z\,1}^*$ must vanish when applying the boundary condition and asymptotic matching. This simplifies equations (21) and (22) to give

$$\boldsymbol{B}_\perp^* \equiv \hat{z} \times \nabla_\perp \psi^* \; , \tag{24}$$

$$\frac{\partial \psi^*}{\partial z} = 0 \; . \tag{25}$$

Now, equation (25) can be integrated to give $\psi^* = \psi^*(\boldsymbol{x}_\perp, t)$. When this is matched asymptotically to the solution in the corona, we have

$$\psi^*(\boldsymbol{x}_\perp, t) = \psi(\boldsymbol{x}_\perp, z = 0, t) \; . \tag{26}$$

This is sufficient to fix ψ^* and, in fact, completes the solution of ψ and φ everywhere. The sequence to solve for φ and ψ is as follows:

From equation (13), we have

$$\nabla_\perp^2 \psi = J \; , \tag{27}$$

where J is an undetermined function. From equation (11), we have

$$\varphi(z) - \varphi(z_0) = \int_{z_0}^{z} \frac{d\ell}{B}\, \partial_t \psi \; , \tag{28}$$

where $d\ell$ is the incremental length along a field line and where both z_0 and z are in the "corona." We can now apply the matching condition in equation (20) to get

$$\varphi_{\text{plate}}^{z=L}(\boldsymbol{x}_\perp, t) - \varphi_{\text{plate}}^{z=0}(\boldsymbol{x}_\perp, t) = \int_{z_0 \to 0}^{z \to L} \frac{d\ell}{B}\, \partial_t \psi \; . \tag{29}$$

Since the left-hand side is completely known, one can, in principle, solve for J from equation (29) and thus find ψ (from eq. [27]). In this manner, we now know ψ and φ (from eq. [28]) in the corona. The layer quantities ψ^* and φ^* are then determined from equations (26) and (19). This concludes the determination of ψ and φ. In particular, we note that the usual procedures are valid, since we have proved that φ and ψ do not indeed vary rapidly along z in the layer.

We now proceed to solve for B_{z_2}, for which the boundary layer cannot be ignored. We take equation (1) to second order in ϵ, operate on it with $\nabla_\perp \cdot$, and use equation (3) to

$O(\epsilon^2)$ to find the equation for $B^*_{z\,2}$:

$$B_0 \nabla^2 B^*_{z\,2} = -\nabla_\perp \cdot (\nabla_\perp \psi^* \nabla^2_\perp \psi^*) , \tag{30}$$

where $\nabla^2 = \nabla^2_\perp + \partial^2_z$. Note that this equation is different from the corresponding equation for B_{z2} in the corona, equation (14). In particular, this equation can be solved subject to the boundary condition $B^*_{z\,2}(\text{plates}) = 0$ and then matched to B_{z2} outside. As shown in the Appendix, equation (14) by itself cannot satisfy line-tied boundary conditions. Thus we have shown that B_{z2} has variations in z on the scale of $L_\perp$, indicating that the line-tied plate problem has, in general, boundary layer structure near the plates with $\partial_z \sim L_\perp^{-1}$.

Finally, we summarize the present results by noting that the set

$$\boldsymbol{B} \cdot \nabla \nabla^2_\perp \psi = 0 , \tag{31}$$

$$\boldsymbol{B} \cdot \nabla \varphi = \partial_t \psi , \tag{32}$$

$$B_0 \nabla^2 B_{z2} = -\nabla_\perp \cdot (\nabla_\perp \psi \nabla^2_\perp \psi) , \tag{33}$$

can be used as a set of equations that is *uniformly* valid throughout the plates. In particular, equations (31) and (32) reduce to equations (25) and (18) as the plates are approached, and equation (33) reduces to equation (14) in the corona and equation (30) in the layer. The set is to be solved subject to $B_{z2} = 0$ at the plates, and $\varphi = \varphi_{\text{plate}}$, as specified by the motions there.

4. SUMMARY

In this paper, we have reexamined the validity of the well-known reduced equations of MHD when they are applied in situations involving "line-tied" boundary conditions. The reduced equations are basically an asymptotic expansion of the full MHD equations using the smallness parameters $L_\perp/L_z \ll 1$ and $|\boldsymbol{B}_\perp| \ll B_z$. A validity test on any asymptotic ordering is that the higher order terms in the expansion be successively smaller as well as be consistent with the boundary conditions. In carrying expansions to higher order, we found that in the presence of line-tying, the next nonzero term in the expansion of B_z—namely, B_{z2}—does not satisfy the boundary conditions. As is well known in asymptotic theory, this immediately suggests the presence of a boundary layer. In this paper, we have shown that a boundary layer indeed exists near the conducting plates and that the thickness of the layer along z is of order $L_\perp \ll L_z$. We have shown that B_{z2} satisfies a different equation in the layer, allowing boundary conditions to be satisfied. We also show that ψ and φ do not satisfy new equations in the layer, at least to lowest order, and, as such, the usual reduced equations for ψ and φ are uniformly valid.

While this result does not have any large bearing on using reduced equations, it is of importance to establish this, since the result is not clearly evident a priori. In particular, a current controversy in solar coronal heating (van Ballegooijen 1985; Antiochos 1987) involves a proof based on reduced equations (van Ballegooijen 1985) in the presence of line-tying: it is important to establish whether or not this proof is affected by the boundary layer. We conclude that it is not.

The identification of the boundary layer, while relatively benign in implication in our present calculation, could take on greater significance if more general orderings were considered. For instance, one could imagine very high "winding numbers" in Parker's (1994) coronal heating scenario so that, while $L_\perp \ll L_z$ may still be dictated based on plate separation and perpendicular flow scales, the ordering $|\boldsymbol{B}_\perp| \sim B_z$ may be more appropriate. Alternatively, the ordering $L_\perp/L_z \ll |\boldsymbol{B}_\perp|/B_z \ll 1$ could also be considered. These orderings are relevant to consider if one assumes that neighboring equilibria are indeed smooth for $L_\perp/L_z \sim |\boldsymbol{B}_\perp|/B_z \ll 1$ based on the calculation of van Ballegooijen (1985).

Another point worth mentioning in connection with the boundary layer is that the presence of the layer does not seem to affect the "deflection" of a field beyond that expected normally. Field line deflection could be measured, for example, by the equation $dx/dz = B_x/B_z$. Considering that $B_z \to B_{z0} + B_{z2}$, with B_{z2} varying on scales of order $L_\perp$, it seems that the deflection, $\Delta x/L_z$, scales at most as $(B_x/B_z)[1 + O(\epsilon)]$. Thus the layer only affects the normal deflection by $O(\epsilon)$, at least within the reduced ordering.

The presence of a boundary layer has also been noted by other authors. Zweibel & Boozer (1985) and later Zweibel & Li (1987) found that layers appear for line-tied boundary conditions with compressible plate motions, i.e., plate motions that cannot be described by a potential φ. The B_z perturbation in these layers is an order larger in our incompressible case; i.e., there is a nonzero B_{z1}. Such layers are to be expected, however, since compressible motions necessarily violate the reduced equations ordering scheme, which is essentially incompressible. More recently, simulations by Mikic, Schnack, & Van Hoven (1990) have clearly shown a scale size near the plates of order $L_\perp$ in the z-direction. Our calculation in this paper is more formal and general, and it shows that even incompressible plate motions lead to boundary layers.

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APPENDIX

Here is a simple example that illustrates the incompatibility of the standard "reduced" ordering of the MHD equations with line-tied boundary conditions. Consider a system consisting of a uniform plasma existing between two perfectly conducting surfaces at $z = 0$, L, respectively. Threading the plasma and the surfaces is a uniform magnetic field $\boldsymbol{B}_0 = B_0 \hat{z}$. The system is in static equilibrium and infinite in extent in the x-y plane. At $t = 0$ let the surface at $z = L$ be subject to small quasistatic, incompressible displacements and deformations so that

$$\tilde{\boldsymbol{u}}_\perp(z = L) = \hat{z} \times \nabla_\perp \tilde{\varphi}_0(\boldsymbol{x}_\perp, t) , \tag{A1}$$

with $\tilde{\varphi}_0(\boldsymbol{x}_\perp, t)$ as a known function. Let the surface at $z = 0$ be held fixed. For this system, the standard "reduced" ordering (§ 2) yields the following equations:

$$\tilde{\boldsymbol{u}}_\perp \equiv \hat{z} \times \nabla_\perp \tilde{\varphi} , \tag{A2}$$

$$\tilde{\boldsymbol{B}}_\perp \equiv \hat{z} \times \nabla_\perp \tilde{\psi} , \tag{A3}$$

$$\frac{\partial \tilde{\psi}}{\partial t} = B_0 \frac{\partial \tilde{\varphi}}{\partial z} , \tag{A4}$$

$$B_0 \frac{\partial}{\partial z} (\nabla_\perp^2 \tilde{\psi}) = 0 , \tag{A5}$$

$$B_0 \nabla_\perp^2 \tilde{\tilde{B}}_z = -\nabla_\perp \cdot (\nabla_\perp \tilde{\psi} \nabla_\perp^2 \tilde{\psi}) . \tag{A6}$$

The boundary conditions dictate that the normal magnetic field, and the tangential electric field must be continuous at the surfaces. This translates to the following boundary conditions:

$$\tilde{\varphi}(z = 0) = 0 , \qquad \tilde{\varphi}(z = L) = \tilde{\varphi}_0(\boldsymbol{x}_\perp, t) ; \tag{A7}$$

$$\tilde{\tilde{B}}_z(z = 0) = 0 , \qquad \tilde{\tilde{B}}_z(z = L) = 0 . \tag{A8}$$

Equation (A5) implies the field-aligned current does not vary along z. Assuming that such a current exists for general $\tilde{\varphi}_0$, i.e., $\nabla_\perp^2 \tilde{\psi} \neq 0$, we have

$$\frac{\partial \tilde{\psi}}{\partial z} = 0 . \tag{A9}$$

This yields, from equation (A4),

$$\frac{\partial^2 \tilde{\varphi}}{\partial z^2} = 0 . \tag{A10}$$

From the boundary condition in equation (A7), the solution of equation (A10) is

$$\tilde{\varphi} = \frac{\tilde{\varphi}_0(\boldsymbol{x}_\perp, t) z}{L} . \tag{A11}$$

Plugging $\tilde{\varphi}$ into equation (A4) gives

$$\tilde{\psi} = \int_0^t dt' \, \frac{\tilde{\varphi}_0(\boldsymbol{x}_\perp, t')}{L} . \tag{A12}$$

Thus we know $\tilde{\varphi}$ and $\tilde{\psi}$ from equations (A11) and (A12); $\tilde{\psi}$ satisfies equations (A9) and (A5), and in general $\nabla_\perp^2 \tilde{\psi} \neq 0$ as expected. So far all is consistent: $\tilde{\varphi}$ varies linearly in z from 0 at $z = 0$ to $\tilde{\varphi}_0(\boldsymbol{x}_\perp, t)$ at $z = L$, and $\tilde{\boldsymbol{B}}_\perp$ has no z variation. This causes tangential discontinuities in the magnetic field at the plates, but this is allowed by our solid plate model and indicates the development of skin currents on the plates.

Now consider equation (A6), which gives the first nonvanishing correction to B_z. The right-hand side is now known and is strictly a function of $\boldsymbol{x}_\perp, t$:

$$\nabla_\perp^2 \tilde{\tilde{B}}_z = f(\boldsymbol{x}_\perp, t) . \tag{A13}$$

The general solution to equation (A13) is

$$\tilde{\tilde{B}}_z = \int_{-\infty}^{\infty} dk [A_\pm(k, z, t) e^{k(x \pm iy)} + C_\pm(k, z, t) e^{k(ix \pm y)}] + \tilde{\tilde{B}}_{z_p}(\boldsymbol{x}_\perp, t) , \tag{A14}$$

where $\tilde{\tilde{B}}_{z_p}$ is the particular solution depending on $f(\boldsymbol{x}_\perp, t)$.

It is clear from the form of equation (A14) that the boundary condition in equation (A8) cannot be applied in general. We are forced to conclude that boundary layers must exist for $\tilde{\tilde{B}}_z$ close to the surfaces. It is reasonable to expect that the ordering assumption $\partial_z \ll \partial_\perp$ must fail in those regions.

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