Stability of Boolean Networks of Arbitrary Topology

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MURI on Non-Linear Dynamics for Sensor Networks

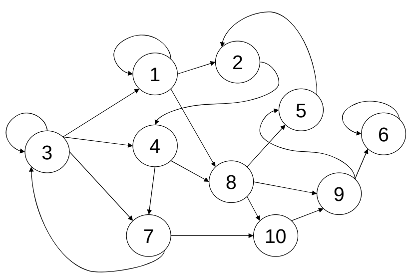
Motivation

•Boolean networks have been extensively studied as a simple model for genetic control.

•Boolean networks realized in electronics exhibit chaotic behavior and could be useful sensor platforms

Under what circumstances do Boolean networks exhibit stable or chaotic dynamics?

Defining Boolean Networks



Input 1	Input 2	Update Value
0	0	1
0	1	0
1	0	1
1	1	0

Above: Sample truth table governing an arbitrary node's evolution p_i = 0.5

Left: Sample Boolean Network, N =10, K = 2

•Boolean networks consist of N nodes, each of which has a value $s_i \in \{0,1\}$ at time step t.

•Traditionally, at each time step, every node i is updated synchronously according to some Boolean function of the values of its input nodes at the previous time.

•More generally, we can consider that time delays exist along links, such that $s_i(t)$ depends on the node values at time $t - \tau_{ij}$.

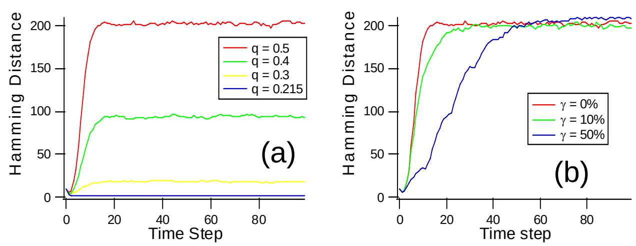
•The Boolean function takes the form of a truth table for each node i; the probability of a one appearing in a randomly chosen entry in the update column is p_i.

•Let the sensitivity $q_i = p_i^2 + (1-p_i)^2$ be the probability that the update at node i changes if a random input to i changes.

Growth of Perturbations

•The distance between two states of a Boolean network is given by the Hamming distance h, which is the number of nodes that differ in their s_i values.

•Boolean networks exhibit stable and 'chaotic' (i.e., unstable) dynamics, where close initial conditions either initially converge or diverge in time



(a) Hamming Distance vs. Time for several different uniform q's, and (b) for q = 0.5 with and without delays along the links (γ of the links have a delay 10, (1-γ) have delay 1).

Steady state behavior is dependent on the sensitivity q, but not on time delays.

Theory for Networks of Arbitrary Topology

Previous work:

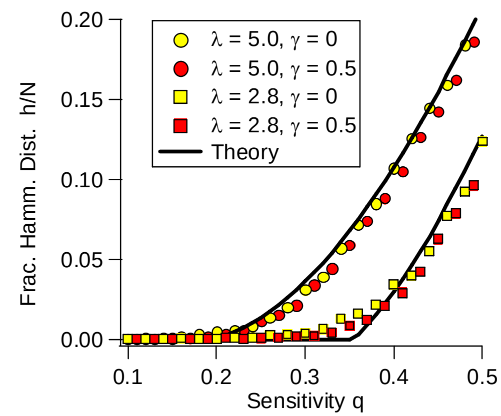
•All nodes have the same q and the topology is completely random.

Our work:

•We allow for arbitrary network topology and non-uniform q_i

•The predicted transition is based on the largest eigenvalue of a modified adjacency matrix

Numerical Tests Agree with Theory



Fractional Hamming Distance h/N vs. q for two different networks with N=1000 and truncated power-law degree distribution (gamma = 2.1). The largest eigenvalue of the adjacency matrices are controlled by the amount of in-degree/out-degree correlation. Each data point is the average of 100 realizations of the truth table, but the same network of connections.

Conclusions

•We have developed a theory that predicts the stability of Boolean networks and the steady-state Hamming distance of small perturbations.

All previous work was restricted to a simple class of random networks. Our work, for the first time, allows consideration of nontrivial network topologies likely to be relevant for the MURI work of Lathrop et al. and Gauthier et al.